The University of Chicago

# SYMMETRIC FUNCTIONS OF INFINITELY MANY ELEMENTS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
CHARLES ARTHUR MESSICK

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## Part I

## Theory of Symmetric Functions

### Introduction

A fundamental problem of symmetric functions is that of expressing a symmetric $\Sigma$ function of one kind in terms of symmetric $\Sigma$ functions of another kind. The already published solutions of the problem seem to be valid only when the number of elements and of sequences is finite, and those solutions require a step by step process of building up from the simplest functions, except when there is just one sequence of elements.[1] The solution given here is valid for an infinity of sequences, each composed of an infinity of elements. A step by step process is not required, and all the finite cases are included as simple special cases. By transforming into series an infinite product each factor of which is an infinite series, formulas are derived for expressing symmetric functions of one kind directly in terms of another kind. The heretofore difficult and tedious problem of expressing any general symmetric $\Sigma$ function in terms of the elementary functions or the sums of powers of the elements is done here by means of a few simple substitutions into the formulas. The various finite cases are obtained by assigning the value zero to all the elements not in the given finite set. In this entire paper no special consideration will be given to a finite number of elements or sequences, except when examples are given to show that such are special cases.

In a polynomial or infinite series which is a symmetric function, if the whole *terms* are the elements, there is nothing

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[1] See Cullis or Junker for the step by step processes. Dresden gives a formula for the problem when there is only one sequence.

The complete titles of all references are listed at the end of the paper.

of interest to consider, but if each element appears as a factor of a term, i.e. with a numerical or literal coefficient, then it is of interest to know whether or not a permutation of the *elements* affects the value of the function. If the number of elements is finite, convergence does not need to be considered. Since each function of the infinite set of elements is an infinite series or product, convergenceconditions and validity of mathematical operations are important topics presented here, which, it seems, have not been discussed elsewhere. Moreover, the method of development permits us to use the formulas to transform many infinite products into infinite series. Examples of a number of different types are given, the resulting series in some cases apparently being new.[2]

Definitions of the elementary symmetric functions, the $p$'s, and the sums of powers of the elements, the $s$'s.

If we have given a set of elements

$$(1)\qquad \begin{array}{cccccccccc} a_1 & b_1 & c_1 & d_1 & \cdots & l_1 & \cdots \\ a_2 & b_2 & c_2 & d_2 & \cdots & l_2 & \cdots \\ a_3 & b_3 & c_3 & d_3 & \cdots & l_3 & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdots \end{array}$$

each row and each column being infinite in length, we can form the following functions:

$$\begin{array}{lllll} p_1 = \Sigma a_i & p_{01} = \Sigma b_i & p_{001} = \Sigma c_i & \cdots & \cdots \\ p_2 = \Sigma a_i a_j & p_{02} = \Sigma b_i b_j & p_{002} = \Sigma c_i c_j & \cdots & \cdots \\ p_3 = \Sigma a_i a_j a_k & p_{03} = \Sigma b_i b_j b_k & p_{003} = \Sigma c_i c_j c_k & \cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots \\ p_{11} = \Sigma a_i b_j & p_{101} = \Sigma a_i c_j & p_{011} = \Sigma b_i c_j & p_{111} = \Sigma a_i b_j c_k & \cdots \\ p_{21} = \Sigma a_i a_j b_k & p_{201} = \Sigma a_i a_j c_k & p_{021} = \Sigma b_i b_j c_k & \cdots\cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \end{array}$$

---

[2] See especially the Theta-series on page 85.

where the summation is taken for $i$, $j$, $\cdots$, ranging from one to infinity, except that no two elements are to be taken from the same row of the set (1) and each possible term occurs only once. For example, the first term of $p_{232}$ is $a_1a_2b_2b_4b_5c_5c_7$.

We can also form the following functions:

| | | | | |
|---|---|---|---|---|
| $s_1 = \sum a_i$ | $s_{01} = \sum b_i$ | $\cdots$ | $s_{11} = \sum a_ib_i$ | $s_{1c1} = \sum a_ic_i$ | $\cdots$ |
| $s_2 = \sum a_i^2$ | $s_{c2} = \sum b_i^2$ | $\cdots$ | $s_{21} = \sum a_i^2b_i$ | $s_{2c1} = \sum a_i^2c_i$ | $\cdots$ |
| $s_3 = \sum a_i^3$ | $s_{03} = \sum b_i^3$ | $\cdots$ | $s_{12} = \sum a_ib_i^2$ | $s_{1c2} = \sum a_ic_i^2$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

In forming any of the terms of the $s$'s, all of the elements are to be taken from the same row of the set (1); that is, the subscripts attached to the elements are all the same in one term. For example, the first term of $s_{242}$ is $a_1^2b_1^4c_1^2$.

For both the $p$'s and the $s$'s the first subscript refers to the elements of the first column, the second subscript refers to the elements of the second column, etc. Each subscript denotes the number of elements to be taken from the column and used as factors of a term. We shall speak of each column of the elements of the set (1) as a sequence. Although each row can be called a sequence because it contains an infinity of elements which must be defined by some law, the name will not be applied to the rows.

The simplest symmetric functions are those formed of the elements of one sequence only, with only a finite number of elements. The definition given in many algebras[3] is essentially as follows: a symmetric function of the elements of a sequence $a_1$, $a_2$, $a_3$, $\cdots$, $a_n$, is a function whose value is not changed by any permutation whatsoever of the elements. In this paper we shall consider functions of *infinitely many sequences of elements, each sequence containing an infinity of elements.* We define a symmetric function

[3] DeComberousse, Niewenglowski, Weber. See also Macmahon. Bocher's definition is more restricted, but he gives the equivalent of our definition in a theorem.

of the elements of the infinite array (1) to be a function whose value is not changed by any permutation whatsoever of the *rows* of elements.[4] This definition implies that if any element bearing the index $j$ is interchanged with the element of the same column bearing the index $k$, then *all* the $j$th elements must be interchanged with the $k$th elements of the corresponding columns. There is no interest in considering permutations of the columns.

From the definition of symmetric function we see that if it contains one term or factor of a given type, it contains all the terms or factors of that type. Then the sum or the product of these terms or factors can be represented by a summation sign $\Sigma$ or a product sign $\Pi$ followed by a typical term or factor. If the function is a sum of several terms, it can be represented by a linear combination of the $\Sigma$'s, since each group of terms can be represented by a $\Sigma$. If the function is a product of several factors, we shall in this paper first express the product as a sum in terms of the $\phi$'s or the $s$'s if this is possible. It is of importance to be able to express any $\Sigma$ function or any product function in terms of the $\phi$'s or the $s$'s if we are to solve the problems stated above.

## Convergence of the s's and the $\phi$'s.

According to the definition of symmetric function, all the $\phi$'s and s's are apparently symmetric functions of the array (1), for if every permutation of the rows is made, these functions remain unchanged in appearance except for changes in the arrangement of the terms; but since each $\phi$ and each $s$ is an infinite series, convergence conditions must be considered in order that the values of the functions shall not be changed by any permutation of the rows of the given array. Every absolutely convergent infinite

---

[4] This wording is based on Cullis, but see also Macmahon, and Pocher. In these works, however, the array is finite in both directions.

series or product is a symmetric function of its terms, or factors, as the case may be. For, an absolutely convergent infinite series or product remains unchanged in value when its terms, or factors, are rearranged in order. Conversely, every symmetric function of infinitely many elements must be absolutely convergent if it is expressed as an infinite series, or product, of its elements. For its value must remain unchanged by every permutation of the rows of elements. A conditionally convergent series or product may be made to have any arbitrary value by rearranging the terms,[5] or the factors;[6] so that a conditionally convergent series or product could not be symmetric. Therefore, absolute convergence and symmetry are equivalent conditions for infinite series or products.

We shall now prove that if $s_1$, $s_{01}$, $s_{001}$, $\cdots\cdots$, are absolutely convergent, then all of the $s$'s and all of the $t$'s are.

Write
$$s_1 = a_1 + a_2 + a_3 + a_4 + \cdots\cdots$$
$$s_1 = a_1 + a_2 + a_3 + a_4 + \cdots\cdots$$

Multiplying term by term,
$$s_2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 + \cdots\cdots .$$

Since each series called $s_1$ is absolutely convergent, then $s_2$ is absolutely convergent; for if each term $a_i$ of an absolutely convergent series is multiplied by a factor $q_i$ such that $|q_i| < K$, a constant, for every $i$, the resulting series is absolutely convergent.[7] In this case $q_i$ is $a_i$, and since $s_1$ converges, $La_i = C$ and therefore satisfies the condition that $|a_i| < K$. Using $s_2$ in the second line in place of $s_1$, we obtain $s_3$ in the same manner. For the same reason, $s_3$ is absolutely convergent. By continuing the process, we obtain $s_1$, $s_2$, $s_3$, $\cdots\cdots$, and in the same manner $s_{01}$, $s_{02}$, $s_{03}$, $\cdots$, $s_{001}$, $s_{002}$, $s_{003}$, $\cdots\cdots$, etc., all of which are absolutely convergent.

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[5] Bromwich, sec. 28. Smail, sec. 141.
[6] Bromwich, sec. 41.
[7] Bromwich, sec. 18. Smail, sec. 126.

Now $s_j = a_1^j + a_2^j + a_3^j + \cdots\cdots$

and $s_{0k} = b_1^k + b_2^k + b_3^k + \cdots\cdots$

so that $s_{jk} = a_1^j b_1^k + a_2^j b_2^k + a_3^j b_3^k + \cdots\cdots$.

Multiplying the last line term by term by a third $s$, and that result by another $s$, etc., we obtain $s_{jk\cdots r}$, which is absolutely convergent for each choice of $j$, $k$, $\cdots$, $r$. Therefore all the $s$'s are absolutely convergent if $s_1$, $s_{01}$, $s_{001}$, $\cdots$ are.

The definitions of the $\phi$'s and $s$'s show that $\phi_1$, $\phi_{01}$, $\phi_{001}$, $\cdots$, are equal respectively to $s_1$, $s_{01}$, $s_{001}$, $\cdots$, so that when the functions of either of these sets are symmetric, so are the functions of the other set.

Let $\phi_1 = a_1 + a_2 + a_3 + \cdots\cdots$ converge absolutely, and write

$$\begin{aligned}\phi_2 = {} & a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + \cdots\cdots \\ & + a_2a_3 + a_2a_4 + a_2a_5 + \cdots\cdots \\ & + a_3a_4 + a_3a_5 + \cdots\cdots \\ & + \cdots\cdots\cdots \\ & \cdots\cdots\cdots = \sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{\infty} a_j .\end{aligned}$$

Each line of this series is absolutely convergent, for it can be written $a_i \sum_{i+1}^{\infty} a_j$, and $\sum_{i+1}^{\infty} |a_j| \leqq \sum_{1}^{\infty} |a_j|$, which is convergent because $\phi_1$ is absolutely convergent. Let A denote $\sum_{1}^{\infty} |a_j|$. Now replace each factor of each term by its absolute value, and we have $\sum_{1}^{\infty} |a_i| \sum_{i+1}^{\infty} |a_j| \leqq \sum_{1}^{\infty} |a_i| \sum_{1}^{\infty} |a_j| = A^2$, from which $|\phi_2| \leqq A^2$, and $\phi_2$ converges absolutely. Similarly we can write $|\phi_3| = |\sum_{1}^{\infty} a_i \sum_{i+1}^{\infty} a_j \sum_{j+1}^{\infty} a_k| \leqq \sum_{1}^{\infty} |a_i| \sum_{1}^{\infty} |a_j| \sum_{1}^{\infty} |a_k| = A^3$, and therefore $\phi_3$ converges absolutely. We can show in the same manner that every $\phi_j$ converges absolutely for every $j$. Now write

$$\begin{aligned}\phi_{11} = {} & \cdot \quad a_1b_2 + a_1b_3 + a_1b_4 + \cdots\cdots \\ & + a_2b_1 \quad \cdot \quad + a_2b_3 + a_2b_4 + \cdots\cdots \\ & + a_3b_1 + a_3b_2 \quad \cdot \quad + a_3b_4 + \cdots\cdots \\ & + a_4b_1 + a_4b_2 + a_4b_3 \quad \cdot \quad + \cdots\cdots \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

Each line of this sum can be written $a_i \sum_1^{\infty}{}' b_j$, where the accent indicates that $b_i$ is omitted from the sum. Let $B = \sum_1^{\infty} |b_j|$. Since $p_{01}$ is absolutely convergent, then $\sum_1^{\infty}{}' |b_j| \leqq B$, and $|p_{11}| \leqq |\sum_1^{\infty} a_i \sum_1^{\infty}{}' b_j|$ $\leqq \sum_1^{\infty} |a_i| \sum_1^{\infty}{}' |b_j| \leqq AB$. From this we find that $p_{11}$ is absolutely convergent. By breaking $p_{jk\cdots q}$ into its component sums and by repeating the process just used, we can show that $p_{jk\cdots q}$ is absolutely convergent for every choice of $j$, $k$, $\cdots$, $q$, provided $p_1$, $p_{01}$, $p_{001}$, $\cdots\cdots$, are absolutely convergent.

We have now proved the following theorem:

THEOREM I. *Given the sequences* $(a_i)$, $(b_i)$, $(c_i)$, $\cdots\cdots$, $(l_i)$, $\cdots$. *A necessary and sufficient condition that all the s's and all the p's shall converge absolutely, or shall be symmetric, is that* $s_1$, $s_{01}$, $s_{001}$, $\cdots\cdots$, *or* $p_1$, $p_{01}$, $p_{001}$, $\cdots\cdots$, *converge absolutely, or are symmetric. Absolute convergence and symmetry are equivalent conditions of the p's and the s's; either one may be replaced by the other.*

Expansion of an infinite product in terms of the $p$'s, and in terms of the $s$'s, and other fundamental formulas.

In the next paragraphs some fundamental formulas will be derived by means of which we can express symmetric functions in terms of the $p$'s or the $s$'s, together with the conditions under which these expressions represent the functions.

Let $x$ be a complex variable, and let $\alpha$, $\beta$, $\gamma$, $\cdots\cdots$, $\lambda$, $\cdots$, be a set of distinct positive integers arranged in ascending order of magnitude. Now form the series

$$v_i(x) = a_i x^{\alpha} + b_i x^{\beta} + c_i x^{\gamma} + \cdots\cdots\cdots + l_i x^{\lambda} + \cdots\cdots, \qquad i = 1, 2, 3, \cdots\cdots .$$

With this set of $v$'s we shall form the infinite product $\Pi = \Pi[1 + v_i(x)]$. Various convergence conditions will be imposed on the $v$'s and $\Pi$ as we proceed. The exponents $\alpha$, $\beta$, $\gamma$, $\cdots$,

will be called the weights of the elements $a_i$, $b_i$, $c_i$, $\cdots\cdots$. The weight will always be denoted by the Greek letter corresponding to the letter representing the element. The weight of a term which is the product of several elements of the same or different sequences is the sum of the weights of the elements making up the term. For example, the weight of each term of $s_2$ is $3\alpha$, of $s_{32}$ is $5\alpha + 2\beta$, and in general, the weight of $s_{jk\cdots q}$ is denoted by $h$, where $h = j\alpha + k\beta + \cdots + q\lambda$. The weight of $p_3$ is also $3\alpha$, of $p_{32}$ is $5\alpha + 2\beta$, and in general, the weight of $p_{jk\cdots q}$ is $h$. If all the terms of a symmetric function have the same weight, the weight of the function is defined to be the weight of any one of its terms. The idea of weights is not essential to the theory, but it simplifies the discussion of the relations between symmetric functions.

In order to develop the formulas of this section, let the elements of the array (1) satisfy the following conditions: all the $p$'s converge, each $v_i(x)$ converges and is not equal to $-1$, and $\sum|v_i(x)|$ is uniformly convergent for a certain range of values of $x$. From these conditions we see that $\Pi$ is absolutely-uniformly convergent. Since $\Pi$ is unchanged in value when the rows of (1) are permuted, it is a symmetric function of the elements. Writing out $\Pi$ we have

$$\begin{aligned}
&[1 + a_1x^\alpha + b_1x^\beta + c_1x^\gamma + \cdots\cdots + l_1x^\lambda + \cdots\cdots] \\
\times\ &[1 + a_2x^\alpha + b_2x^\beta + c_2x^\gamma + \cdots\cdots + l_2x^\lambda + \cdots\cdots] \\
\times\ &[1 + a_3x^\alpha + b_3x^\beta + c_3x^\gamma + \cdots\cdots + l_3x^\lambda + \cdots\cdots] \\
\times\ &\cdots\cdots\cdots

Taking 1 from each factor except two, we get all the terms

$$\Sigma a_i a_j x^{2\alpha} + \Sigma b_i b_j x^{2\beta} + \Sigma c_i c_j x^{2\gamma} + \cdots\cdots + \Sigma a_i b_j x^{\alpha+\beta} + \Sigma a_i c_j x^{\alpha+\gamma} + \Sigma b_i c_j x^{\beta+\gamma} \cdots\cdots$$

$$= \phi_2 x^{2\alpha} + \phi_{02} x^{2\beta} + \phi_{002} x^{2\gamma} + \cdots\cdots\cdots + \phi_{11} x^{\alpha+\beta} + \phi_{101} x^{\alpha+\gamma} + \phi_{011} x^{\beta+\gamma} \cdots\cdots .$$

Continuing this process we get each one of the $\phi$'s exactly once, multiplied by the associated power of $x$, and we have[8]

$$\Pi = 1 + \phi_1 x^{\alpha} + \phi_2 x^{2\alpha} + \phi_{01} x^{\beta} + \phi_3 x^{3\alpha} + \phi_{11} x^{\alpha+\beta} + \phi_{001} x^{\gamma} + \phi_4 x^{4\alpha}$$

$$+ \phi_{21} x^{2\alpha+\beta} + \phi_{02} x^{2\beta} + \phi_{101} x^{\alpha+\gamma} + \cdots\cdots .$$

Associated with $\phi_{jk\cdots q}$ is a power of $x$ whose exponent is $j\alpha + k\beta + \cdots\cdots + q\lambda$, that is, the weight of $\phi$. Since we may have $2\alpha = \beta$, $\alpha + \beta = \gamma$, etc., the coefficients of some of the powers of $x$ may be grouped together. After this is done, the expansion of the product may be written

$$\Pi = 1 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots\cdots + C_w x^w + \cdots\cdots . \tag{2}$$

In this form, some of the $C$'s may be zero, for not all of the powers of $x$ need be present. The coefficient $C_w$ of $x^w$ in the series (2) is the sum of all the terms containing $x^w$ in the expansion of the product $\Pi$. No other terms of weight $w$ are to be found in the series. Therefore, in terms of the $\phi$'s, $C_w$ is the sum of all of the $\phi$'s of weight $w$. Each of the $\phi$'s is a symmetric function, and therefore $C_w$ is a symmetric function, and it is of weight $w$. As a general formula we can write

$$C_w = \Sigma \phi_{jk\cdots q} , \tag{3}$$

where the summation is taken for all values of the indices such that $j\alpha + k\beta + \cdots\cdots + q\lambda = w$ .

Since the weights of the $\phi$'s depend upon $\alpha$, $\beta$, $\gamma$, $\cdots\cdots$, the representation of the $C$'s in terms of the $\phi$'s also depends upon $\alpha$, $\beta$, $\gamma$, $\cdots\cdots$ . The weight of $\phi_{jk\cdots q}$ is $j\alpha + k\beta + \cdots\cdots + q\lambda$ , where $\alpha$, $\beta$, $\cdots\cdots$, $\lambda$, $j$, $k$, $\cdots\cdots$, $q$, are all integers. In order to write down the representation of the $C$'s in terms of the $\phi$'s so that it

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[8] See Knopp, p. 487, sec. 253, Satz 1, for proof of this. The terms in the series are arranged in the order in which they would appear if $\alpha$, $\beta$, $\cdots\cdots$, were 1, 2, $\cdots\cdots$ .

has a meaning for all choices of the weights $\alpha$, $\beta$, $\gamma$, $\cdots$, $\lambda$, $\cdots$ another notation will be used. Denote $\phi_{jk\cdots q}$ by $\phi[jk\cdots q]$. Let $r$, $s$, $\cdots$, $t$, be chosen so that $r/\alpha = j$, $s/\beta = k$, $\cdots$, $t/\lambda = q$. In this notation $\phi_{jk\cdots q} = \phi[\frac{r}{\alpha}\frac{s}{\beta}\cdots\frac{t}{\lambda}]$. This puts the weights in evidence in the symbol. For any choice of the integers $\alpha$, $\beta$, $\cdots$ $\lambda$, $\cdots$, $r$, $s$, $\cdots$, $t$, the symbol has no meaning if any one of the numbers $r/\alpha$, $\cdots$, $t/\lambda$, does not reduce to an integer. Also, the weight is $j\alpha + k\beta + \cdots + q\lambda = \frac{r}{\alpha}\alpha + \frac{s}{\beta}\beta + \cdots + \frac{t}{\lambda}\lambda = r + s + \cdots + t = w$. Writing the $C$'s in terms of the $\phi$'s,

$$C_1 = \phi[\tfrac{1}{\alpha}],$$

$$C_2 = \phi[\tfrac{2}{\alpha}] + \phi[\tfrac{1}{\alpha}\tfrac{1}{\beta}],$$

$$\cdots\cdots\cdots\cdots,$$

$$C_w = \Sigma\,\phi[\tfrac{r}{\alpha}\tfrac{s}{\beta}\cdots\tfrac{t}{\lambda}], \tag{3'}$$

where $r/\alpha$, $s/\beta$, $\cdots$ $t/\lambda$, are integers and $r + s + \cdots + t = w$.

THEOREM II. *Given the array* (1) *such that the $\phi$'s and $v$'s converge, and $v_i(x) \neq -1$, and $\Sigma|v_i(x)|$ converges uniformly. Then* (2) *represents* $\Pi$, *where the $C$'s are given by* (3) *or* (3').

We shall next transform the product $\Pi = \Pi[1 + v_i(x)]$ into a power series $1 + C_1x + C_2x^2 + \cdots + C_wx^w + \cdots$, obtaining the formula for $C_w$ expressed in terms of the $s$'s. We wish to find the least restrictive conditions under which the transformation can be performed; for this reason no hypothesis shall be stated at the beginning and no condition shall be imposed until needed.

We shall use the following method. First we shall define $u = \log \Pi$, so that $u = \Sigma \log[1 + v_i] = \Sigma\,[v_i - \frac{1}{2}v_i^2 + \frac{1}{3}v_i^3 \cdots]$. Each power of $v_i$ is a function of $x$, and after we find these powers of $v_i$ we shall arrange $\log[1 + v_i]$ according to powers of $x$, obtaining $\log[1 + v_i(x)] = \sum_h u_{ih}x^h$. Summing with respect to $i$ we can obtain $\sum_i \sum_h u_{ih}x^h = u$, but by interchanging the order of summation we obtain $u$ expressed as a power series in which the coefficients

are functions of the $s$'s defined on page 3. Next we shall set $\Pi = e^{u}$, and expand this according to powers of $u$. Each power of $u$ is to be arranged according to powers of $x$, and then $e^{u}$ is to be arranged according to powers of $x$, the coefficients $C_w$ being functions of the $s$'s.

In order to begin the work outlined above, we must have each $v_i(x)$ converge and $|v_i(x)| \leqq 1$ but $v_i(x) \neq -1$. We shall suppose that these conditions are satisfied.

Consider now the logarithm of the $i$th factor of $\Pi$;

$$(4) \qquad \log [1 + v_i(x)] = v_i - \tfrac{1}{2} v_i^2 + \tfrac{1}{3} v_i^3 - \cdots\cdots + \frac{(-1)^{n-1}}{n} v_i^n \cdots\cdots .$$

By using the multinomial theorem we obtain

$$\frac{(-1)^{n-1}}{n} v_i^n = \frac{(-1)^{n-1}}{n} (a_i x^{\alpha} + b_i x^{\beta} + \cdots\cdots + l_i x^{\lambda} + \cdots\cdots)^n$$

$$= \frac{(-1)^{n-1}}{n} \sum \frac{n!}{j!k!\cdots r!} (a_i x^{\alpha})^j (b_i x^{\beta})^k \cdots\cdots (l_i x^{\lambda})^r$$

$$(5) \qquad = \sum \frac{(-1)^n m!}{J!!} a_i^{\,j} b_i^{\,k} \cdots\cdots l_i^{\,r} x^h ,$$

where for abbreviation we define $J!! = j!k!\cdots r!$, $m = n - 1$, and $j\alpha + k\beta + \cdots\cdots + r\lambda = h$, and where the summation is taken only for the non-negative values of $j$, $k$, $\cdots$, $r$, which satisfy the equation $j + k + \cdots\cdots + r = n$. We see that $j$, $k$, $\cdots$, $r$, are not all zero at once. We note that $J!!^q = j!^q k!^q \cdots r!^q$.

Since (4) is a convergent power series, it converges absolutely and uniformly[9] for $|v_i| < 1 - \varepsilon$, where $\varepsilon$ is an arbitrary positive number less than 1. Let $R_1$ denote the radius of a circle such that $|v_i(x)| < 1 - \varepsilon$ when $|x| < R_1$ and $i = 1, 2, 3, \cdots$, so that the series (4) is absolutely and uniformly convergent when $|x| < R_1$. We now impose the condition on the elements of the array (1) that they be chosen so that $R_1$ shall be greater than zero. Since $v_i(x)$

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[9] Bromwich, sec. 50. Smail, sec. 165, 169. Tannery, sec. 184.

is a convergent power series in $R_1$, it is also absolutely convergent, and therefore $v_i^n$ converges absolutely in the same region.[10] Therefore the terms of (5) can be arranged according to powers of $x$, for each value of $n$, and the resulting power series is absolutely convergent. After this rearrangement is made, (4) becomes a uniformly convergent series in $R_1$, every term of which is a power series convergent in $R_1$. A theorem by Weierstrass states that such a series can be arranged according to powers of $x$, the coefficients of the various terms of a given power being added to form a single coefficient of that power. The resulting series converges absolutely for $|x| < R_1$. In making this last rearrangement, we find that in the terms of (4) the same power of $x$ may appear in several powers of $v_i$. To collect all the terms containing $x^h$, we must pick from the several powers of $v_i$; that is, we allow $n$ to take on several values while $h$ is kept fixed. In picking out all the terms having $x^h$, we get $x^h[\Sigma \frac{(-1)^m m!}{j!\,!} a_i^j b_i^k \cdots l_i^r]$, the summation being taken for the values of $j, k, \cdots, r$, which satisfy the equation $j\alpha + k\beta + \cdots + r\lambda = h$, (here having a changing $m$ which by definition is $m = j + k + \cdots + r - 1$). Let $u_{ih}$ denote the expression inside the brackets.

Now we can write $\log [1 + v_i] = u_{i1}x + u_{i2}x^2 + u_{i3}x^3 + \cdots\cdots$. Since this is a convergent power series, each $u_{ih}$ represents a finite quantity. There are as many terms in $u_{ih}$ as there are solutions of the equation $j\alpha + k\beta + \cdots + r\lambda = h$. This shows that $u_{ih}$ has a finite number of terms for each finite $h$. Now write

$$
\begin{aligned}
u &= \Sigma\, i\log[1 + v_i] \\
&= u_{11}x + u_{12}x^2 + u_{13}x^3 + \cdots\cdots + u_{1h}x^h + \cdots\cdots \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + u_{i1}x + u_{i2}x^2 + u_{i3}x^3 + \cdots\cdots + u_{ih}x^h + \cdots\cdots \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{6}
$$

[10] Bromwich, sec. 27. Smail, sec. 146, 165.

Referring to the definitions of the $s$'s we see that $\sum_i u_{ih}$ is a function of the $s$'s. The series (6) as it now is can be written $\sum_i \sum_h u_{ih} x^h$ and must be changed to $\sum_h \sum_i u_{ih} x^h$ in order to reduce it to a power series whose coefficients are functions of the $s$'s. Some conditions which permit the interchange of order of summation are:

1. Let $\sum_h u_{ih} x^h$ converge in $R_1$ and $\Sigma \log[1 + v_i(x)]$ converge uniformly in $R_2 < R_1$. (The Weierstrass theorem[11] requires this condition, and states that then $\sum_h \sum_i u_{ih} x^h$ converges in $R_1$ and is equal to $\sum_i \sum_h u_{ih} x^h$.)

2. Let $\sum_h u_{ih} x^h$ converge in $R_1$ and $\sum_i |v_i(x)|$ converge uniformly in $R_1$. (If $\sum_i |v_i(x)|$ converges uniformly, then so does $\Sigma \log[1 + v_i(x)]$. The conclusion follows from 1 above.)[12]

3. Let $\sum_h |u_{ih} x^h| = U_i$ converge, and let $\sum_i U_i$ converge. (This is Cauchy's theorem.[13])

4. Let $\sum_i u_{ih} x^h$, $\sum_h u_{ih} x^h$, and $\sum_{ih} u_{ih} x^h$, all converge.[14]

5. Let $\sum_{ih} |u_{ih} x^h|$ converge.[15]

If the elements of the array (1) are chosen so that any one of these five conditions is satisfied, then we can sum the series (6) by columns. Because of the conditions already imposed on page 11 on the elements of the array (1), conditions 1, 2, and 4 are already partly satisfied, for $\sum_h u_{ih} x^h$ converges and equals $\log[1+v_i]$. If we proceed under one of these three conditions the least restrictive additional conditions would be needed; but 2 is stronger than 1, since if we have 2 satisfied then 1 is also satisfied, but not conversely. For definiteness we shall proceed under condition 1,

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[11] Harkness and Morley, sec. 81. Knopp, sec. 249, p. 480.

[12] Pierpont,II, sec. 209, 138. Smail, sec. 258, 257.

[13] Bromwich, sec. 82. Smail, sec. 232, 224.

[14] Bromwich, sec. 80. Smail, sec. 218.

[15] Bromwich, sec. 82. Smail, sec. 230.

and impose the condition on the array (1) that the elements be such that $\Sigma\log[1 + v_i(x)]$ converge uniformly. This is equivalent to the condition that $\Pi$ converge uniformly. [16] Using the definition of $u_{ih}$ and the definitions of the $s$'s, we see that provided all of the $s$'s converge

$$\sum_i u_{ih} = \sum_i \Sigma \frac{(-1)^m m!}{j!i!} a_i^j b_i^k \cdots l_i^r = \Sigma \sum_i \frac{(-1)^m m!}{j!i!} a_i^j b_i^k \cdots l_i^r = \Sigma \frac{(-1)^m m!}{j!i!} s_{jk\cdots r} ,$$

where the unmarked $\Sigma$ denotes summation for those values of the indices which satisfy the equation $j\alpha + k\beta + \cdots + r\lambda = h$, and where $m = j + k + \cdots + r - 1$. Since $u_{ih}$ has a finite number of terms, as shown before, the change of order of the repeated sums is permissible if the $s$'s converge. The convergence of the $s$'s does not follow from the convergence of $\sum_i u_{ih}$. We must now impose the condition that the $s$'s converge.

Now we define

$$u_h = x^h \Sigma \frac{(-1)^m m!}{j!i!} s_{jk\cdots r} = x^h \sum_i u_{ih} , \tag{7}$$

where the summation is taken for the values of the indices which satisfy the equation $j\alpha + k\beta + \cdots + r\lambda = h$. With this definition we can write

$$u = \sum_i \log[1 + v_i(x)] = \sum_h u_h ,$$

and the last series converges absolutely, since it is a power series.

$$\text{Since } \Pi = e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^\sigma}{\sigma!} + \cdots ,$$

it is necessary to find expressions for these terms.

Let $u_{h_1}, u_{h_2}, \cdots$, denote any of the terms of the series $u = \Sigma u_h$, where $h_1, h_2, \cdots$, are distinct and denote a set of numbers selected from 1, 2, 3, $\cdots$. By using the multinomial theorem

$$\frac{u^\sigma}{\sigma!} = \Sigma \frac{1}{n!\theta!\cdots\rho!} u_{h_1}^n u_{h_2}^\theta \cdots u_{h_q}^\rho , \tag{8}$$

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[16] Smail, sec. 267.

where the summation is taken for the values of the exponents such that $\eta + \theta + \cdots + \rho = \sigma$. Each factor is power of a series of the form (7). Taking the first,

$$(9) \qquad u_{h_1}^{\eta} = \left[ x^{h_1} \sum \frac{(-1)^m m!}{j_1! \cdots} s_{jk\cdots r} \right]^{\eta}$$

$$= \sum \frac{\eta!}{\eta_1! \eta_2! \cdots \eta_i!} \left[ \frac{(-1)^{m_1} m_1!\, x^{h_1}}{j_1!\cdots} s_{j_1 k_1 \cdots r_1} \right]^{\eta_1} \cdots \left[ \frac{(-1)^{m_i} m_i!\, x^{h_i}}{j_i!\cdots} s_{j_i \cdots r_i} \right]^{\eta_i}$$

where the last summation is taken for values of the $\eta$'s such that $\eta_1 + \eta_2 + \cdots + \eta_i = \eta$. The other powers of the other $u$'s are similar to (9). The product $u_{h_1}^{\eta} u_{h_2}^{\theta} \cdots u_{h_q}^{\rho}$ is formed by taking the sum of all the products formed by taking one term from each of the series like (9) as factors. Every term of $\frac{u^{\sigma}}{\sigma!}$ is then a product of certain $s$'s with the proper coefficient. Let $\pi$ be any one of the numbers $\eta_1, \eta_2, \cdots, \eta_i, \theta_1, \theta_2, \cdots, \cdots, \rho_1, \rho_2, \cdots, \rho_i$. For each $(s_{jk\cdots r})^{\pi}$ in a term of the product, there is in its coefficient the factor $\frac{(-1)^{m\pi} m!^{\pi} x^{\pi h}}{\pi!\, j!!^{\pi}}$. The $\pi!$ in the numerator of (9) is cancelled by the denominator of (8), and likewise for $\theta! \cdots \rho!$. In each of the expressions like (9) the exponent is separated into parts such that $\eta_1 + \eta_2 + \cdots + \eta_i = \eta$, $\theta_1 + \theta_2 + \cdots = \theta$, $\cdots\cdots$, $\rho_1 + \rho_2 + \cdots = \rho$; but in (8) it is required that $\eta + \theta + \cdots + \rho = \sigma$. Therefore, in the product it is required that

$$\eta_1 + \eta_2 + \cdots + \theta_1 + \theta_2 + \cdots + \rho_1 + \rho_2 + \cdots + \rho_k = \sigma;$$

or, in other terms, $\pi_1 + \pi_2 + \cdots + \pi_q = \sigma$. Now we obtain

(10)

$$\frac{v^{\sigma}}{\sigma!} = \sum x^{w} \frac{(-1)^{\sum m_i \pi_i} m_1!^{\pi_1} m_2!^{\pi_2} \cdots m_q!^{\pi_q}}{\pi_1! \pi_2! \cdots \pi_q!\, j_1!!^{\pi_1} \cdots j_q!!^{\pi_q}} (s_{j_1 k_1 \cdots r_1})^{\pi_1} \cdots\cdots (s_{j_q k_q \cdots r_q})^{\pi_q},$$

where the summation is taken for all values of the indices which satisfy the equation $\pi_1 + \pi_2 + \cdots + \pi_q = \sigma$, and where by definition $w = \sum_i \pi_i h_i$, and $h_i = j_i \alpha + k_i \beta + \cdots + r_i \lambda$. These last operations can be performed to obtain (10) because $u$ is a power series (and

absolutely convergent) and therefore $\frac{u^\sigma}{\sigma!}$ is a power series converging absolutely for $|x| < R_2$, and uniformly for $|x| \leq R_3 < R_2$.

Now $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\cdots + \frac{u^\sigma}{\sigma!} + \cdots\cdots$ is a series of power series satisfying the Weierstrass theorem, so that it may be arranged according to powers of $x$. After this rearrangement is made the resulting series is convergent in $R_2$. The expression for the coefficient of $x^w$ will be denoted by $C_w$, and is found by taking the sum of all the terms, chosen from $e^u$, such that $\Sigma \pi_i h_i = w$. These terms having been obtained by the use of (10), the sum of the coefficients of $x^w$, that is, $C_w$, is

(11)
$$C_w = \Sigma \frac{(-1)^{\Sigma w_i \pi_i}\, w_1!^{\pi_1} w_2!^{\pi_2} \cdots w_q!^{\pi_q}}{\pi_1!\, \pi_2! \cdots \pi_q!\, J_1!^{\pi_1} \cdots J_q!^{\pi_q}} (s_{j_1 k_1 \cdots r_1})^{\pi_1} \cdots\cdots (s_{j_q k_q \cdots r_q})^{\pi_q},$$

where the summation is to be taken for all the values of the indices for which $\sum_i \pi_i h_i = w$; and where by definition, $h_i = j_i\alpha + k_i\beta + \cdots + r_i\lambda$, and $w_i = j_i + k_i + \cdots + r_i - 1$, and $J!! = j!k! \cdots r!$

Using this expression for $C_w$, we have

$$\Pi[1 + v_i(x)] = 1 + C_1 x + C_2 x^2 + \cdots\cdots + C_w x^w + \cdots\cdots.$$

The series for $e^u$ converges for all values of $u$, but since we know only that $u$ converges for $|x| < R_2$, then $R_2$ is less than or equal to the radius of convergence of the series representation of $\Pi$. We have proved the following theorem:

THEOREM IIIa. *Let the elements of the array* (1) *be given such that* $\Pi[1 + v_i(x)]$ *converges uniformly when* $0 < |x| < R_1$, *where* $v_i(x)$ *is defined on page 7, and where* $|v_i(x)| < 1 - \varepsilon$, $(0 < \varepsilon < 1)$; *and let* (1) *also be given so that all the s's converge. Then for* $|x| \leq R_2 < R_1$ *the product* $\Pi$ *and the series* $1 + \Sigma C_w x^w$ *are equivalent, where the C's are given by formula* (11).

The theorem can be stated with a different hypothesis:

THEOREM IIIb. *Let the elements of the array* (1) *be given such that* $|v_i(x)| < 1 - \varepsilon$, $(0 < \varepsilon < 1)$, *where* $v_i(x)$ *satisfies the definition*

*given on page 7, and let any one of the conditions on page 13 be satisfied, and let all the s's converge. Then* $\Pi[1+v_i(x)] = 1+\Sigma C_w x^w$ *where the C's are given by formula* (11).

As a special case, let $v_i(x) = a_i x$. If $\Sigma a_i = s_1$ is absolutely convergent, then all the s's converge absolutely, by Theorem I; if the $a$'s are real numbers and if $s_1$ and $s_2$ converge, then all the s's converge, and except for $s_1$, the convergence is absolute.[17] In either of these cases, $\Sigma v_i(x)$ converges for all values of $x$, and equals $s_1 x$. For every finite R, the convergence is uniform for $|x| < R$. Now all the conditions of Theorem IIIa are satisfied if $|a_i x| < 1-\varepsilon$. The $C_w$'s become $t_w$'s and formula (11) takes the more simple form[18]

$$(11') \qquad C_w \equiv t_w = \Sigma \frac{(-1)^{\Sigma(j_i-1)\pi_i}}{\pi_1!\cdots\pi_q!\; j_1!^{\pi_1}\cdots j_q!^{\pi_q}} s_{j_1}^{\pi_1}\cdots\cdots s_{j_q}^{\pi_q},$$

where the summation is taken for the values of the indices which satisfy the equation $\Sigma\pi_i j_i = w$.

Fundamental Theorem of Symmetric Function Theory.

The hypothesis of Theorem IIIb permits the choice of one of the conditions on page 13. It was shown on pages 11-12 that if $|v_i| < 1-\varepsilon$, then $\Sigma_h v_{ih} x^h$ converges in $R_1$, so that if we choose condition 2, we have added to the hypothesis of Theorem IIIb only that $\Sigma|v_i(x)|$ converge uniformly. As mentioned on pages 13-14, this condition implies that $\Pi[1+v_i(x)]$ is absolutely-uniformly convergent in $R_1$. Therefore the value of $\Pi$ is not changed by any permutation of the rows of the array (1) and is a symmetric function of

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[17] Bromwich, sec. 18. Smail, sec. 126.

[18] A derivation of formula (11') is given for a finite number of elements, with a proof by means of symmetric functions, in Serret, Vol. I. sec. 198-201. The formula is given with suggestions for deriving it, in Macmahon, Vol.I. Chap.I, and in Macmahon, Intro. Chap.I, but only for a finite number of elements. Formula (11) is given in Macmahon, Vol.II, p. 282-3 for the special case of two sequences.

the elements of (1). Theorem IIIb requires the convergence of all the $s$'s. According to Theorem I all the $s$'s converge and are symmetric functions of the elements of (1) if $s_1$, $s_{01}$, $s_{001}$, ····, converge absolutely. All the $t$'s also converge and are symmetric functions of the elements of (1), so that if the conditions stated in this and the preceding paragraph are satisfied, the hypothesis of Theorem II is also satisfied. Therefore we have proved

THEOREM IV. *Let the elements of the array* (1) *be given such that* $|v_i(x)| < 1-\varepsilon$, $(C<\varepsilon<1)$, *when* $0<|x|<R_1$, *and* $\Sigma|v_i(x)|$ *converges uniformly in* $R_1$, *where* $v_i(x)$ *is as defined of page 7, and such that* $s_1$, $s_{01}$, $s_{001}$, ····, *converge absolutely. Then* $\Pi[1+v_i(x)]$ *converges absolutely-uniformly in* $R_1$ *and equals* $1+\Sigma C_w x^w$, *where*

$$
\begin{aligned}
&C_1 = t_1 = s_1 ,\\
&C_2 = t_2 + t_{01} = \tfrac{1}{2!}[(s_1^2 - s_2) + (2s_{01})] ,\\
&C_3 = t_3 + t_{11} + t_{001} = \tfrac{1}{3!}[(s_1^3 - 3s_1s_2 + 3s_3) + (6s_1s_{01} - 6s_{11}) + (6s_{001})],\\
&C_4 = t_4 + t_{21} + t_{01} + t_{101} + t_{0001} = \tfrac{1}{4!}[(s_1^4 - 6s_1^2s_2 + 8s_1s_3 - 6s_4 + 3s_2^2)\\
&\qquad + (12s_1^2s_{01} - 24s_1s_{11} + 24s_{21} - 12s_2t_{01}) + (12s_{01}^2 - 12s_{01})\\
&\qquad + (24s_1s_{001} - 24s_{101}) + (24s_{0001})] .
\end{aligned}
\tag{12}
$$

*In general the* $C$'s *are given by formula* (3) *and formula* (11) *and* II, *the* $t$'s, *the* $s$'s, *and the* $C$'s *are symmetric functions of the elements of the array* (1).

This is the fundamental theorem of the theory of symmetric functions. In the remaining portion of Part I it will be supposed that the conditions of this theorem are satisfied.

When the weights $\alpha$, $\beta$, $\gamma$, ····, are distinct positive integers as required in the definition of $v_i(x)$, all the choices of these weights give essentially the same results. When the weights are the set $\alpha=1$, $\beta=2$, $\gamma=3$, ····, we shall say "the set 1, 2, 3,···"; and for the set 1, 2, 3, ···, we obtain the $C$'s as given by formulas (12) above. If now, for example, we wish to write the expressions

for the $C$'s for the set 1, 3, 4, ····, instead of setting $\beta = C$ as we might be inclined to do, or $\beta = 3$, $\gamma = 4$, etc., we set all the $b$'s equal to zero. This causes all the $\phi$'s and $s$'s to be zero in which the second subscript shows that some of the $b$'s are factors of each term; that is, if the second subscript is different from zero, the $\phi$ or $s$ is zero. Therefore the expressions given in (12) may be used, and they may be shortened by omitting every term in which there is a $\phi$ or $s$ whose second subscript is not zero. This also allows us to use unchanged the formula for the weights of the functions, namely $h = j\alpha + k\beta + \cdots + r\lambda$. A similar discussion could be given for any set of weights satisfying the definition of $v_i(x)$, so that we may take the weights to be 1, 2, 3, ····, by defining some of the elements of the array (1) to be zero if necessary.

Expressions for the $\phi$'s in terms of the $s$'s,
and for the $s$'s in terms of the $\phi$'s.

We shall now show how to obtain the expression for $\phi_{jk\cdots r}$ in terms of the $s$'s.

Form the array of elements (1') using the sequences $\{Y_\alpha a_i\}$, $\{Y_\beta b_i\}$, ·····, $\{Y_\lambda l_i\}$, ····, where $Y_\alpha$ is a variable parameter having the same value for all the $a$'s at the same time, $Y_\beta$ one having the same value for all the $b$'s at the same time, etc. Let the elements of the array (1') be such that when $Y_\alpha = Y_\beta = \cdots = 1$, the conditions of Theorem IV are satisfied. Using the elements (1') form the $v$'s as defined on page 7, and form the infinite product $\Pi[1 + v_i(x)]$. By Theorem IV we now obtain the $C$'s as functions of $Y_\alpha a_i$, $Y_\beta b_i$, ····, $Y_\lambda l_i$, ····. Letting $\phi'$ and $s'$ denote the $\phi$'s and $s$'s for (1'), we obtain[19]

$$\phi'_{jk\cdots r} = Y_\alpha^j Y_\beta^k \cdots Y_\lambda^r \phi_{jk\cdots r} \quad \text{and} \quad s'_{jk\cdots r} = Y_\alpha^j Y_\beta^k \cdots Y_\lambda^r s_{jk\cdots r} .$$

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[19] Bromwich, sec. 18. Smail, sec. 74.

If we write out a set of equations like (12), for the elements of (1'), but where each $\phi'$ and $s'$ is written out in terms of its elements according to the definitions on pages 2 and 3, each equation is of the form $C_w = \varphi(Y,\phi) = \psi(Y,s)$, which is an identity in the $Y$'s, and $\varphi$ and $\psi$ are both polynomials in the $Y$'s (because $j\alpha + k\beta + \cdots + r\lambda = w$ has a finite number of solutions). This identity holds for $C \leqq Y \leqq 1$, since $\phi'$ and $s'$ are absolutely convergent when $Y = 1$. From formula (3) and the relation of $\phi'$ to $\phi$ we see that every term of $\varphi(Y,\phi)$ has the form $Y_\alpha^j Y_\beta^k \cdots Y_\lambda^r \phi_{jk\ldots r}$ where $j\alpha + k\beta + \cdots + r\lambda = w$. Formula (11) shows that in $\psi(Y,s)$ there are terms of the form $A(Y_\alpha^{j_1} Y_\beta^{k_1} \cdots Y_\lambda^{r_1} s_{j_1 k_1 \cdots r_1})^{\pi_1} \cdots (Y_\alpha^{j_i} \cdots Y_\lambda^{r_i} s_{j_i \cdots r_i})^{\pi_i}$ where A depends on the indices, and where the indices satisfy the equation $\Sigma\pi_i h_i = \Sigma\pi_i(j_i\alpha + k_i\beta + \cdots + r_i\lambda) = (\Sigma\pi_i j_i)\alpha + (\Sigma\pi_i k_i)\beta + \cdots = w$. Since we must use all values of the indices which satisfy this condition, there is at least one set of numbers such that $\Sigma\pi_i j_i = j$, $\Sigma\pi_i k_i = k$, $\cdots\cdots$, $\Sigma\pi_i r_i = r$, where $j$, $k$, $\cdots$, $r$, are those given with a certain $\phi_{jk\ldots r}$. Therefore there is in $\psi(Y,s)$ a term having $Y_\alpha^j Y_\beta^k \cdots Y_\lambda^r$ with a coefficient expressed in terms of $s$'s, for each $Y_\alpha^j Y_\beta^k \cdots Y_\lambda^r \phi_{jk\ldots r}$ existing in $\varphi(Y,\phi)$. Since the polynomial $\varphi$ equals the polynomial $\psi$ identically in the $Y$'s, the coefficient of a power of the $Y$'s in $\varphi$ equals identically the coefficient of the same power of the $Y$'s in $\psi$.[20] Therefore each $\phi$ may be expressed in terms of $s$'s, since each $\phi$ occurs just once in $\varphi$, and we have proved the following

THEOREM V. *The function $\phi_{jk\ldots r}$ can be expressed in terms of the s's by a formula derived from equations* (12), *viz.*

$$\phi_{jk\ldots r} = \sum \frac{(-1)^{\Sigma\pi_i\pi_i}\, \pi_1!^{\pi_1} \cdots \pi_i!^{\pi_i}}{\pi_1! \cdots \pi_i!\, j_1!^{\pi_1} \cdots j_i!^{\pi_i}} (s_{j_1 k_1 \ldots r_1})^{\pi_1} \cdots (s_{j_i k_i \ldots r_i})^{\pi_i}, \tag{13}$$

[20] Bocher, sec. 2, definition on page 5, and Theorem 2, page 6. Also Niewengloski, p. 12.

*where the summation is taken for the values of the indices which satisfy the equations* $\Sigma\pi_i j_i = j$, $\Sigma\pi_i k_i = k$, $\cdots\cdots$, $\Sigma\pi_i r_i = r$, *and where* $m_i = j_i + k_i + \cdots + r_i - 1$.

In equations (12) the terms are grouped in parentheses so that the first, second, third, etc. group represents the first, second, third, etc. $\phi$ respectively. The first group in each equation is the well know expression for $\phi_j$ in terms of s's, which is a special case of (13) where $b_i = c_i = \cdots\cdots = 0$.

In order to call special attention to the fact, we state the following

COROLLARY V.1. *The representation of* $\phi_{jk\ldots r}$ *in terms of the s's is independent of the weights* $\alpha$, $\beta$, $\gamma$, $\cdots$.

Another interesting point to notice is that the coefficient of the product of s's in (13) is symmetric in the m's, $\pi$'s, and $\sigma$'s, so that if $j$, $k$, $\cdots$, $r$, are permuted, the coefficients of the expression in s's are unchanged. Only the subscripts are permuted. For example,

$$\phi_{21} = \tfrac{1}{2}(s_1^2 s_{01} - 2s_1 s_{11} + 2s_{21} - s_2 s_{01}),$$

$$\phi_{12} = \tfrac{1}{2}(s_1 s_{01}^2 - 2s_{01}s_{11} + 2s_{12} - s_1 s_{02}),$$

$$\phi_{012} = \tfrac{1}{2}(s_{01}s_{001}^2 - 2s_{001}s_{011} + 2s_{012} - s_{01}s_{002}).$$

COROLLARY V.2. *Let* $j'$, $k'$, $\cdots$, $r'$, *be a permutation of* $j$, $k$, $\cdots$, $r$. *The expressions for* $\phi_{jk\ldots r}$ *and for* $\phi_{j'k'\ldots r'}$ *have the same coefficients, and the indices of one expression may be obtained from the other by permutation.*

We shall now derive some relations between the s's and C's, and expressions for the s's in terms of the $\phi$'s. Suppose the conditions of Theorem IV are satisfied, so that

$$\Pi = 1 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots\cdots .$$

Then
$$u = \log \Pi = (\Pi - 1) - \tfrac{1}{2}(\Pi - 1)^2 + \tfrac{1}{3}(\Pi - 1)^3 \cdots\cdots$$

$$= \Sigma \frac{(-1)^{n-1}}{n} (C_1 x + C_2 x^2 + C_3 x^3 + \cdots\cdots)^n .$$

The series for $\Pi - 1$ is absolutely convergent when $|x| < R_1$, and

therefore the $n$th power converges absolutely,[21] and can be arranged according to powers of $x$. In order to have $u$ converge absolutely and uniformly, we must determine $R_2 < R_1$ such that $|\Pi - 1| < 1 - \varepsilon$, $(0 < \varepsilon < 1)$, when $|x| < R_2$. From the multinomial theorem

$$\frac{(-1)^{n-1}}{n}(C_1x + C_2x^2 + \cdots)^n = \sum \frac{(-1)^m m!}{j!\,!}(C_1x)^j(C_2x^2)^k \cdots (C_ix^i)^r$$

$$= \sum \frac{(-1)^m m!}{j!\,!} x^{j+2k+\cdots+ir} C_1^{\,j} C_2^{\,k} \cdots C_i^{\,r},$$

where the summation is taken for the values of the indices such that $j + k + \cdots + r = n \ (= m + 1)$. Now each term of $u$ is absolutely convergent and the series for $u$ converges uniformly. Therefore, by the Weierstrass theorem, the series can be arranged according to powers of $x$. Picking out all the terms having a given power, $h$, of $x$, we have

$$u_h = x^h \sum \frac{(-1)^m m!}{j!\,!} C_1^{\,j} C_2^{\,k} \cdots C_i^{\,r}, \tag{14}$$

where $j + 2k + \cdots + ir = h$, $m = j + k + \cdots + r - 1$, and $u = \Sigma u_h$.

Using (14) $\quad u = C_1x + x^2(-\tfrac{1}{2}C_1^2 + C_2) + x^3(\tfrac{1}{2}C_1^3 - C_1C_2 + C_3) + \cdots$.

Using (7) $\quad u = sx + x^2(-\tfrac{1}{2}s_2 + s_{01}) + x^3(\tfrac{1}{2}s_3 - s_{11} + s_{001}) + \cdots$,

if we take $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\cdots$. Since each of these series is a power series representing the same function for all values of $|x| < R_2$, the coefficients of the same powers in the two series are identically equal.[22]

THEOREM VI. *If the elements of the array* (1) *satisfy the hypothesis of Theorem IV, then for each fixed integer* $h$,

$$\sum \frac{(-1)^m m!}{j!\,!} s_{jk\ldots r} = \sum \frac{(-1)^m m!}{j!\,!} C_1^{\,j} C_2^{\,k} \cdots C_i^{\,r},$$

*where on the left* $j\alpha + k\beta + \cdots + r\lambda = h$, *and on the right* $j + 2k + \cdots + ir = h$, *and where* $m = j + k + \cdots + r - 1$.

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[21] Smail, sec. 146.

[22] Bromwich, sec. 52. Smail, sec. 177.

This equality is true whether $\alpha$, $\beta$, $\gamma$, $\cdots\cdots$, are 1, 2, 3,$\cdots$ or other numbers, but if they are other numbers the coefficients in the second of the two equations given above the theorem must be changed to correspond. The coefficients in terms of the $C$'s will remain unchanged. In the special case where $b_i = c_i = \cdots\cdots = 0$, all of the $p$'s and $s$'s are zero except $p_j$ and $s_j$. Then we obtain the well known formulas

$$s_1 = p_1\,, \quad s_2 = p_1^2 - 2p_2\,, \quad s_3 = p_1^3 - 3p_1p_2 + 3p_3\,, \quad \cdots\cdots\cdots\,.$$

To obtain $w$ and $w_h$ in terms of $p$'s, we substitute formula (3') into (14). We use (3') because the $C$'s are not independent of $\alpha$, $\beta$, $\gamma$, $\cdots\cdots$. Since $C_n = \Sigma\, p[\frac{r}{\alpha}\,\frac{s}{\beta}\cdots\frac{t}{\lambda}]$, where $r + s + \cdots + t = w$, by the multinomial theorem

$$C_{w_1}^{\,j} = \Sigma \frac{j!}{j_1!j_2!\cdots j_{i_1}!}\left\{p[\frac{r_1}{\alpha}\cdots\cdots\frac{t_1}{\lambda}]\right\}^{j_1}\cdots\cdots\left\{p[\frac{r_{i_1}}{\alpha}\cdots\cdots\frac{t_{i_1}}{\lambda}]\right\}^{j_{i_1}}\,,$$

where $r_n + \cdots\cdots + t_n = w_1$ for $n = 1,2,\cdots$, $i_1$ and where $j_1 + j_2 + \cdots + j_{i_1} = j$. Then $j_1r_1 + \cdots + j_1t_1 + \cdots\cdots + j_{i_1}r_{i_1} + \cdots + j_{i_1}t_{i_1}$

$$= j_1w_1 + \cdots\cdots + j_{i_1}w_1 = jw_1\,.$$

Now the product $\dfrac{(-1)^m\,m!}{J!!}\,C_1^{\,j}\,C_2^{\,k}\cdots\cdots C_K^{\,n}$

$$= (-1)^m\,m!\sum\frac{\left\{p[\frac{r_1}{\alpha}\cdots\cdots\frac{t_1}{\lambda}]\right\}^{j_1}\cdots\cdots\left\{p[\frac{r_{i_1}}{\alpha}\cdots\cdots\frac{t_{i_1}}{\lambda}]\right\}^{j_{i_1}}}{j_1!\quad\cdots\cdots\quad j_{i_1}!}\cdots\cdots\cdots$$

$$\times\sum\frac{\left\{p[\frac{r_K}{\alpha}\cdots\cdots\frac{t_K}{\lambda}]\right\}^{n_1}\cdots\cdots\left\{p[\frac{r_{i_K}}{\alpha}\cdots\cdots\frac{t_{i_K}}{\lambda}]\right\}^{n_{i_K}}}{n_1!\quad\cdots\cdots\quad n_{i_K}!}\,,$$

where $([j_1r_1 + \cdots + j_1t_1] + \cdots\cdots + [j_{i_1}r_{i_1} + \cdots + j_{i_1}t_{i_1}]) + \cdots\cdots$

$$+ ([n_1r_K + \cdots + n_1t_K] + \cdots\cdots + [n_{i_K}r_{i_K} + \cdots + n_{i_K}t_{i_K}])$$

$$= (j_1\cdot 1 + \cdots\cdots + j_{i_1}\cdot 1) + \cdots\cdots\cdots + (n_1\cdot K + \cdots\cdots + n_{i_K}\cdot K)$$

$$= (1\cdot j) + (2\cdot k) + \cdots\cdots + (K\cdot n) = h$$

from the condition on (14). Since *each* $C$ contains only a finite

number of terms, each sum in the parentheses is a polynomial. Multiplying these polynomials, any term of the result is formed by taking as a factor one term from each of the parentheses. Every term of the result then is a product of factors of the type $\frac{1}{\pi_i!}\left\{\phi\left[\frac{r_i}{\alpha}\cdots\frac{t_i}{\lambda}\right]\right\}^{\pi_i}$ with a condition on the indices, and finally multiplied by $(-1)^m m!$ . The condition on the indices is that $\Sigma(\pi_i r_i + \cdots + \pi_i t_i) = h$ . With (14) we had by definition $m = j + k + \cdots + n - 1$ . Since we must have $j_1 + \cdots + j_{i_1} = j$ we see that the sum of a certain first group of $\pi$'s must equal $j$; similarly the sum of a second group must equal $k$; etc. Therefore $\pi_1 + \pi_2 + \cdots + \pi_i - 1 = m$ . In (14) we take the sum of all terms which satisfy the condition $j + 2k + \cdots + ir = h$, and therefore now take the sum of all terms satisfying the equivalent condition $\Sigma\pi_i(r_i + \cdots + t_i) = h$ .

THEOREM VII. *If the elements of the array* (1) *satisfy the hypothesis of Theorem IV and* $|\Pi - 1| < 1 - \varepsilon$ *when* $|x| < R_2$, *then* $\log \Pi = \Sigma u_h$ *where*

$$(15) \qquad u_h = x^h \sum \frac{(-1)^m m!}{\pi_1! \cdots \pi_i!} \left\{\phi\left[\frac{r_1}{\alpha}\cdots\frac{t_1}{\lambda}\right]\right\}^{\pi_1} \cdots\cdots \left\{\phi\left[\frac{r_i}{\alpha}\cdots\frac{t_i}{\lambda}\right]\right\}^{\pi_i},$$

*where the summation is taken for all values of the indices for which* $\Sigma\pi_i(r_i + \cdots + t_i) = h$, *and where by definition* $\pi_1 + \cdots + \pi_i - 1 = m$.

Using formula (7) we obtain $u$ as a power series which converges for $|x| < R_1$. Using (15) we obtain $u$ as a power series which converges for $|x| < R_2 \leq R_1$. Then for $|x| < R_2$ the series are identically equal. When the $s$'s and $\phi$'s are expressed in terms of the elements of (1), this equality is an identity in the elements. Now if we replace the elements of (1) by those of (1') as described in the proof of Theorem V on page 20, and collect the coefficients of each power of the $Y$'s the equation is an identity in the $Y$'s, and each side of the identity is a polynomial in the $Y$'s. By

referring to (7) we see that on one side of the identity each term has the form

$$\frac{(-1)^{j+k+\cdots+q-1}\ (j+k+\cdots+q-1)!}{j!\ k!\ \cdots\ q!}\ \Gamma_\alpha^{\,j}\ \Gamma_\beta^{\,k}\cdots\Gamma_\lambda^{\,q}\ s_{jk\ldots q}\ ,$$

where $j\alpha + k\beta + \cdots + q\lambda = h$. In the definition on page 10 of the symbol $\phi[\frac{r_1}{\alpha}\cdots\frac{t_1}{\lambda}]$, it is required that $r_1/\alpha = j_1$, $\cdots$, $t_1/\lambda = q_1$ where $j_1, \cdots, q_1$, must be integers such that $j_1$ is the number of $\alpha$'s, etc. Then for the array (1'), $\left\{\phi[\frac{r_1}{\alpha}\cdots\frac{t_1}{\lambda}]\right\}^{\pi_1}$ has with it $\Gamma_\alpha^{\frac{\pi_1 r_1}{\alpha}}\cdots\Gamma_\lambda^{\frac{\pi_1 t_1}{\lambda}}$ and the product of $\phi$'s expressed in (15) has with it a $\Gamma_\alpha$ with the exponent $\frac{\pi_1 r_1}{\alpha}+\frac{\pi_2 r_2}{\alpha}+\cdots+\frac{\pi_i r_i}{\alpha}$, multiplied by the other $\Gamma$'s with their corresponding exponents. In other words the terms on the second side of the equation have the form

$$A\ \Gamma_\alpha^{\Sigma\pi_\kappa j_\kappa}\ \Gamma_\beta^{\Sigma\pi_\kappa k_\kappa}\cdots\Gamma_\lambda^{\Sigma\pi_\kappa q_\kappa}\ (\phi_{j_1 k_1\ldots q_1})^{\pi_1}\cdots(\phi_{j_i k_i\ldots q_i})^{\pi_i}\ ,$$

where the sums are taken for $\kappa = 1$ to $\kappa = i$, where A depends on the indices, and where $\alpha\Sigma\pi_\kappa j_\kappa + \beta\Sigma\pi_\kappa k_\kappa + \cdots + \lambda\Sigma\pi_\kappa q_\kappa = h$. Since we must use all the values of the indices which satisfy this condition, there is at least one set of numbers such that $\Sigma\pi_\kappa j_\kappa = j$, $\Sigma\pi_\kappa k_\kappa = k$, $\cdots$, $\Sigma\pi_\kappa q_\kappa = q$, where $j, k, \cdots, q$ are those given on the first side of the equation with a certain $s_{jk\ldots q}$. Therefore in the second side of the identity there is a term in $\Gamma_\alpha^{\,j}\ \Gamma_\beta^{\,k}\cdots\Gamma_\lambda^{\,q}$ whose coefficient is expressed in $\phi$'s, for each $\Gamma_\alpha^{\,j}\ \Gamma_\beta^{\,k}\cdots\Gamma_\lambda^{\,q}$ existing on the first side. Since it is an identity in the $\Gamma$'s, the coefficients of the same power on the two sides are identically equal.

THEOREM VIII. *The function $s_{jk\ldots q}$ can be expressed in terms of the $\phi$'s by a formula derived from* (7) *and* (15), *viz.*

(16)
$$s_{jk\ldots q} = \frac{j!\ k!\ \cdots\ q!}{(-1)^{j+k+\cdots+q-1}\ (j+k+\cdots+q-1)!}\ \Sigma\ \frac{(-1)^m\ m!}{\pi_1!\cdots\pi_i!}\ (\phi_{j_1\ldots q_1})^{\pi_1}\cdots(\phi_{j_i\ldots q_i})^{\pi_i}\ .$$

*where the summation is taken for values of the indices which*

*satisfy the equations* $\Sigma\pi_\kappa j_\kappa = j$, $\cdots$, $\Sigma\pi_\kappa q_\kappa = q$, *and where* $m = \pi_1 + \pi_2 + \cdots + \pi_i - 1$.

As with Theorem V, we have two corollaries.

COROLLARY VIII.1. *The representation of* $s_{jk\ldots q}$ *in terms of the* $p$'*s is independent of the weights.*

COROLLARY VIII.2. *Let* $j'$, $k'$, $\cdots$, $q'$ *be a permutation of* $j$, $k$, $\cdots$, $q$. *The expressions for* $s_{jk\ldots q}$ *and for* $s_{j'k'\ldots q'}$ *have the same coefficients, and the indices of one expression may by obtained from the other by permutation.*

Products and quotients of infinite products.

We shall now consider the expansion of the product $M = \Pi[1 + v_i'] \Pi[1 + v_i'']$, which is the product of two infinite products of the same kind, where $v_i'$ and $v_i''$ are not derivatives but are two unrelated $v$'s. Let $s'$ and $s''$ denote the corresponding $s$'s, and $u'$ and $u''$ the corresponding $u$'s. From (7),

$$\log M = \log\Pi[1 + v_i'] + \log\Pi[1 + v_i'']$$

$$= \Sigma u_h' + \Sigma u_h'' = \sum_h \Sigma \frac{x^h(-1)^m m!}{J!} s'_{jk\ldots r} + \sum_h \Sigma \frac{x^h(-1)^m m!}{J!} s''_{jk\ldots r},$$

where the inner summation is taken for the values of the indices satisfying the equation $j\alpha + k\beta + \cdots + r\lambda = h$, and where $m$ denotes $j + k + \cdots + r - 1$, as previously defined.

Since the series are power series and therefore absolutely convergent, coefficients of the same power of $x$ may be combined. Now we will let $u_h = u'_h + u''_h$; hence $u_h = x^h \Sigma\frac{(-1)^m m!}{J!}(s'_{jk\ldots r} + s''_{jk\ldots r})$, the summation being for indices such that $j\alpha + k\beta + \cdots + r\lambda = h$. This is formula (7) with each $s$ replaced by $s' + s''$, so that throughout the rest of the work of deriving the formula for $C_w$ the same change is to be made. Therefore (11) gives $C_w$ for $M$ by replacing each $s$ by $s' + s''$.

It is obvious that (11) gives $C_w$ for the infinite product

$\Pi[1 + v_i'] \Pi[1 + v_i''] \cdots\cdots \Pi[1 + v_i^{(\mu)}]$ by replacing each $s$ by $s' + s'' + \cdots + s^{(\mu)}$; and for $[\Pi(1 + v_i)]^{\mu}$ by replacing each $s$ by $\mu s$.

For the quotient $Q = \dfrac{\Pi[1 + v_i']}{\Pi[1 + v_i'']}$ we know that $\log Q = \log \Pi[1 + v_i'] - \log \Pi[1 + v_i'']$ so that $\log Q$ has $s' - s''$ where each $s$ occurs in (7).

THEOREM IX. *For an expression of the form*

$$\frac{\Pi[1 + v_i'] \Pi[1 + v_i''] \cdots\cdots \Pi[1 + v_i^{(\mu)}]}{\Pi[1 + v_i^{(\mu+1)}] \cdots\cdots\cdots \Pi[1 + v_i^{(\mu+\nu)}]},$$

*each $s$ in the formulas is replaced by* $(s' + \cdots + s^{(\mu)}) - (s^{(\mu+1)} + \cdots + s^{(\mu+\nu)})$.

For $[\Pi(1 + v_i)]^{\mu}$ each $s$ is replaced by $\mu s$, and for $\dfrac{1}{[\Pi(1 + v_i)]^{\nu}}$ each $s$ is replaced by $-\nu s$. Therefore, for a positive or negative $n$, the $C$'s of the expansion of $[\Pi(1 + v_i)]^{n}$ are given by (11) by replacing each $s$ by $ns$.

For the case of $n = -1$, denote the coefficients by $C_r$ ($r$ suggested by "reciprocal") and the formula is

$$(17) \qquad C_r = \sum \frac{(-1)^{\Sigma(\pi_i + 1)\pi_i}\, \pi_1!^{\pi_1} \cdots \pi_q!^{\pi_q}}{\pi_1! \cdots \pi_q!\, J_1!!^{\pi_1} \cdots J_q!!^{\pi_q}} (s_{j_1 k_1 \cdots r_1})^{\pi_1} \cdots\cdots (s_{j_q k_q \cdots r_q})^{\pi_q},$$

where $m_i = j_i + k_i + \cdots + r_i - 1$, $h_i = j_i\alpha + k_i\beta + \cdots + r_i\lambda$, $J_i!! = j_i! \cdots r_i!$ and where the summation is taken for the indices such that $\Sigma \pi_i h_i = r$. Using this formula we obtain

$$\frac{1}{\Pi[1 + v_i(x)]} = 1 - s_1 x + \frac{x^2}{2!}[(s_1^2 + s_2) - 2s_{01}] - \frac{x^3}{3!}[(s_1^3 + 3s_1 s_2 + 3s_3) - (6s_1 s_{01} + 6s_{11}) + (3s_{001})] \cdots\cdots\cdots.$$

When $n = -1$ and $v_i(x) = \alpha_i x$, formula (17) reduces to

$$(17') \qquad C_r = (-1)^r \sum \frac{1}{\pi_1! \cdots \pi_q!\, j_1^{\pi_1} \cdots j_q^{\pi_q}} s_{j_1}^{\pi_1} \cdots\cdots s_{j_q}^{\pi_q},$$

where the summation is taken for the indices such that $\Sigma \pi_i j_i = r$, since $m_i + 1 = j_i$ and $j_i = h_i$, and $\dfrac{\pi_i!}{J_i!!} = \dfrac{(j_i - 1)!}{j_i!} = \dfrac{1}{j_i}$.

The most general Σ functions in terms of the s's, and in terms of the φ's.

In order to complete the problem of expressing any symmetric Σ function in terms of the φ's or the s's, it will be convenient to use the following notation:

$$(a_i b_i c_i \cdots) = (abc\cdots)_i \ ,$$

$$P_1(1) = \Sigma a_i \ , \qquad P_{11}(1,2) = \Sigma a_i a_j^2 \ , \qquad P_{111}(1,2,3) = \Sigma a_i a_j^2 a_k^3 \ ,$$

$$P_{21}(2,01) = \Sigma a_i^2 a_j^2 b_k \ , \qquad P_{22}(31,12) = \Sigma (a^3b)_i \ (a^3b)_j \ (ab^2)_k \ (ab^2)_l \ ,$$

$$P_{111}(2,01,12) = \Sigma a_i^2 b_j (ab^2)_k \ , \qquad P_{112}(2,01,12) = \Sigma a_i^2 b_j (ab^2)_k \ (ab^2)_l \ .$$

The numbers between commas in the parentheses are the exponents of a set of letters $(abc\cdots)$ in the order given. Each set between commas is used like an "element" in the sum P. The subscripts refer to these "elements" with the same meaning that they have in connection with the φ's. Suppose for example in $P_{22}$ above, we let $(a^3b)_i = T_i$ , and $(ab^2)_i = T_i'$. With this substitution, $P_{22} = \Sigma T_i T_j T_k' T_l'$ ; that is, the sum of terms made up of two elements of the first kind and two of the second kind, each taken from a different row.[28] Compare with $\phi_{22} = \Sigma a_i a_j b_k b_l$ .

We define a set of S's corresponding to the P's:

$$S_{21}(2,01) = \Sigma (a_i^2)^2 (b_i) \ , \qquad S_{22}(31,12) = \Sigma (a^3b)_i^{\,2} (ab^2)_i^{\,2} \ ,$$

$$S_{112}(2,01,12) = \Sigma (a^2)_i (b)_i \ (ab^2)_i^{\,2} \ .$$

It is evident that these S's reduce immediately to the s's, and that is what makes them valuable.

Now if we expand into a series the product

$$\Pi[1 + (a^{\varepsilon_{11}} b^{\varepsilon_{12}} c^{\varepsilon_{13}} \cdots)_i x + (a^{\varepsilon_{21}} b^{\varepsilon_{22}} c^{\varepsilon_{23}} \cdots)_i x^2 + (a^{\varepsilon_{31}} b^{\varepsilon_{32}} c^{\varepsilon_{33}} \cdots)_i x^3 + \cdots] \ ,$$

in which we suppose that the quantities in parentheses satisfy the

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[28] The $T$'s are used here in the same sense in which Cullis uses them to denote the "type" of the sum. Macmahon uses the parenthesis notation. My notation is a combination of the two, (and I think it is easier to comprehend and easier to work with).

conditions imposed on the elements in Theorem IV, we obtain a series of the form $1 + C_1x + C_2x^2 + C_3x^3 + \cdots$, where the $C$'s are expressed in terms of P's or S's by replacing the $p$'s in formula (3) or the $s$'s in formula (11). This is more clearly seen if we write $\Pi[1 + T_i'x + T_i''x^2 + T_i'''x^3 + \cdots]$, where the $T$'s denote the above quantities in parentheses; and $C_1 = \Sigma T_i' = P_1 = S_1$, $C_2 = \Sigma(T_i'T_j') + \Sigma T_i'' = P_2 + P_{01} = \frac{1}{2}[(S_1^2 - S_2) + (2S_{01})]$, etc. From formula (13) we obtain the expression of any P in terms of S's, which may then be expressed in terms of $s$'s. The P's represent symmetric functions of the most general $\Sigma$ type, so that a linear sum of these P's represents any symmetric $\Sigma$ function of the most general form. By use of the P's and S's we can therefore express any symmetric $\Sigma$ function in terms of the $s$'s. Each of the $s$'s can be expressed in terms of the $p$'s, and therefore we can in two steps express any symmetric $\Sigma$ function in terms of the $p$'s. Without any other proof we can state the last theorem.

THEOREM X. *Every symmetric $\Sigma$ function of any number of infinite sequences $(a_i)$, $(b_i)$, $(c_i)$, $\cdots$, satisfying the convergence conditions of Theorem I and the conditions on the $T$'s above, can be expressed in terms of the $s$'s or the $p$'s, the expressions being found by means of the formulas as described above.*

Examples:

(1) $\Sigma a_i^2 b_i = \Sigma(a^2)_i(b)_j = P_{11}(2,01)$.

By formula (13), $P_{11} = S_1S_{01} - S_{11}$.

By definition $S_1 = \Sigma(a^2)_i = s_2$, $S_{01} = \Sigma b_i = s_{01}$, $S_{11} = \Sigma(a^2)_i(b)_i = s_{21}$.

Therefore $\Sigma a_i^2 b_j = P_{11}(2,01) = S_1S_{01} - S_{11} = s_2s_{01} - s_{21}$.

By formula (16) $s_2 = p_1^2 - 2p_2$, $s_{21} = p_1^2p_{01} - p_2p_{01} + p_{21} - p_1p_{11}$.

Therefore $\Sigma a_i^2 b_j = (p_1^2 - 2p_2)p_{01} - (p_1^2p_{01} - p_2p_{01} + p_{21} - p_1p_{11})$

$= p_1p_{11} - p_2p_{01} - p_{21}$.

(2) $\Sigma a_i^2 a_j^2 b_k = P_{21}(2,01) = \frac{1}{2}[S_1^2S_{01} - S_2S_{01} - 2S_1S_{11} + 2S_{21}]$

$= \frac{1}{2}[s_2^2s_{01} - s_4s_{01} - 2s_2s_{21} + 2s_{41}]$

$$= \tfrac{1}{2}[(p_1^2 - 2p_2)^2 p_{01} - (p_1^2 + 2p_2^2 - 4p_1^2 p_2 - 4p_4 + 4p_1 p_3) p_{01}$$
$$- 2(p_1^2 - 2p_2)(p_1^2 p_{01} - p_2 p_{01} + p_{21} - p_1 p_{11}) + 2(p_1^4 p_{01} - p_1^2 p_{11}$$
$$- p_3 p_{11} + p_4 + p_1^2 p_{21} - 3p_1^2 p_2 p_{01} + p_2^2 p_{01} - p_2 p_{21} - p_1 p_{31} +$$
$$2p_1 p_2 p_{01} + 2p_1 p_2 p_{11}$$
$$= 2p_4 p_{01} + p_2 p_{21} - p_0 p_{31} - p_3 p_{11} + p_{41} .$$

(3) In case the letters of only sequence appear in the sum, the P's have only one index number between the commas in the parentheses. For example, $\Sigma a_i a_j^2 = P_{11}(1,2) = S_1 S_{01} - S_{11}$ .

Here $S_1 = \Sigma a_i$, $S_{01} = \Sigma a_i^2$, $S_{11} = \Sigma a_i^3$ .

Therefore $\Sigma a_i a_j^2 = s_1 s_2 - s_3 = p_1 p_2 - 3p_3$ .

Similarly $\Sigma a_i a_j a_k^2 = P_{21}(1,2) = \tfrac{1}{2}[S_1^2 S_{01} - S_2 S_{01} - 2S_1 S_{11} + 2S_{21}]$

$$= \tfrac{1}{2}[s_1^2 s_2 - s_2^2 - 2s_1 s_3 + 2s_4$$
$$= p_1 p_3 - 4p_4 .$$

Sums of this type are special cases of the most general form given above.[24]

The method used in this paper has an advantage over the step by step processes (see Cullis, and Junker) because *any* $\Sigma$ function may be expressed in terms of s's without knowledge of expressions of lower orders. The expression for P in terms of the S's can be written down directly from formula (13), and the S's can be transformed immediately into s's in accordance with their definitions, which can then be expressed in terms of $p$'s by use of formula (16). A formula for P in terms of the $p$'s could be derived, but it is complicated and not of special value. The easier way is to substitute for each s in terms of $p$'s after P has been expressed in terms of s's.

As an example of the use of the preceding theorems, we shall express in terms of s's the set of functions called the $h$'s

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[24] A. Dresden solves this special problem by another method. See Annals of Mathematics, series 2, Vol. 24, p. 227, and Vol. 25, p. 71.

and the set called the $k$'s.[25] The $h$'s are defined by the equation

$$\frac{1}{\Pi[1-a_i x]\ \Pi[1-b_i y]\ \Pi[1-c_i z]\ \cdots\cdots\ \Pi[1-l_i \zeta]} =$$

$$(1 + h_1 x + h_2 x^2 + \cdots)\ (1 + h_{01} y + h_{02} y^2 + \cdots)\ \cdots\cdots\ (1 + h_{0\cdots 1}\zeta + h_{0\cdots 2}\zeta^2 + \cdots)$$

$$= 1 + h_1 x + h_{01} y + h_{001} z + \cdots + h_{0\cdots 1}\zeta + h_2 x^2 + h_{11} xy + h_{101} xz + h_{011} yz + \cdots$$

$$+ h_{02} y^2 + h_{0\cdots 2} z^2 + \cdots + h_3 x^3 + h_{21} x^2 y + h_{12} x y^2 + \cdots + h_{111} xyz + \cdots\cdots .$$

From the expansion of the second member and by comparison with the third, it follows that

$$h_{11} = h_1 h_{01},\quad h_{21} = h_2 h_{01},\quad \cdots\cdots\quad h_{pq} = h_p h_{0q},\quad \cdots\cdots\quad h_{ij\cdots q} = h_i h_{0j} \cdots h_{00\cdots q} .$$

Considering that the letters $x$, $y$, $z$, $\cdots\cdots$ in the formula for the reciprocal are changed to $-x$, $-y$, $-z$, $\cdots\cdots$ in the definition of the $h$'s, the expressions (17') for $C_{r_1}$ can be changed to correspond by multiplying by $(-1)^r$, making the signs always positive. Then

$$\frac{1}{\Pi[1-a_i x]} = (1 + h_1 x + h_2 x^2 + \cdots\cdots) = 1 - C_{11} x + C_{21} x^2 \cdots\cdots + (-1)^r C_{r1} x^r + \cdots,$$

where the C's are given in terms of the $s$'s by formula (17'). By making similar changes, we find similar expressions for $h_{0j}$, $h_{00k}$, $\cdots$. By multiplying these expressions together we obtain

$$(18)\qquad h_{ij\cdots r} = \sum \frac{s_{i_1}^{\pi_1}\ s_{i_2}^{\pi_2} \cdots\cdots s_{0j_1}^{\tau_1} \cdots\cdots s_{0\cdots r_1}^{\omega_1} \cdots s_{0\cdots r_q}^{\omega_q}}{\pi_1! i_1^{\pi_1}\ \pi_2! i_2^{\pi_2} \cdots\cdots \tau_1! j_1^{\tau_1} \cdots\cdots \omega_1! r_1^{\omega_1} \cdots\cdots \omega_q! r_q^{\omega_q}} ,$$

where the summation is taken for the indices which satisfy the equations, $\pi_1 i_1 + \pi_2 i_2 + \cdots\cdots = i$ , $\tau_1 j_1 + \tau_2 j_2 + \cdots\cdots = j$ , $\cdots\cdots$ ,

$\omega_1 r_1 + \omega_2 r_2 + \cdots\cdots = r$ .

Using this formula we obtain

$$h_1 = s_1 ,\quad h_2 = \tfrac{1}{2}(s_1^2 + s_2),\quad h_3 = \tfrac{1}{3!}(s_1^3 + 3 s_1 s_2 + 3 s_3),\quad h_{21} = \tfrac{1}{2}(s_1^2 s_{01} + s_2 s_{01}).$$

By comparison with formula (13), we find that $h_{ij\cdots r}$ is a sum which can be selected from $t_{ij\cdots r}$ by making the signs all positive. It is to be notices first that each $s$ has only one subscript not

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[25] Macmahon, Vol. II, p. 283, 284.

zero, and when this is the case the numerical coefficient of (13) becomes the same as that of (18); so that all the terms of (18) are included in (13), except for sign. Therefore we can obtain $h_{ij\cdots r}$ by selecting all the terms of $f_{ij\cdots r}$ in which each $s$ has only one subscript not zero, making all the signs positive.

The $k$'s are defined by the equation

$$\frac{1}{\Pi[1 - a_i x - b_i y - \cdots]} = 1 + k_1 x + k_{01} y + k_2 x^2 + k_{11} xy + k_{02} y^2 + \cdots\cdots .$$

This is a reciprocal of the form $\frac{1}{\Pi[1 - v_i(x)]}$ where $y = x^2$, $z = x^3$, etc.

By using formula (17) and noticing the negative signs of $a_i$, $b_i$, $\cdots$,

when $r = 1$, $k_1 = s_1$ ;

when $r = 2$, $k_2 + k_{01} = \frac{1}{2!}[(s_1^2 + s_2) + (2s_{01})]$ ;

when $r = 3$, $k_3 + k_{11} + k_{001} = \frac{1}{3!}[(s_1^3 + 3s_1 s_2 + 3s_3) + (6s_1 s_{01} + 6s_{11}) + (6s_{001})]$;

etc.

The definition of the $k$'s is equivalent to saying that $k_{jn\cdots r}$ is the sum of all terms which contain as factors $j$ of the $a$'s, $n$ of the $b$'s, etc. Then if $k_{jn\cdots r}$ is expressed in terms of the $s$'s, the indices of the term containing $(s_{j_1 n_1 \cdots r_1})^{\pi_1} \cdots (s_{j_q n_q \cdots r_q})^{\pi_q}$ must satisfy the conditions $\Sigma \pi_i j_i = j$, $\Sigma \pi_i n_i = n$, $\cdots$, $\Sigma \pi_i r_i = r$, the same conditions which belong to formula (13). Now if we develop the above reciprocal in terms of the $s$'s by using (17) and supposing $y = x^2$, $z = x^3$, etc., all the terms contain products of $s$'s of the type just mentioned, so that every term belongs to one of the $k$'s. The numerical coefficient given by (17) is the same as that of (13) except for sign. In the above reciprocal, every $a$, $b$, $c$, etc. has with it a minus sign, so that every $(s_{jn\cdots r})^{\pi}$ in (17) must be multiplied by $(-1)^{\pi(j + n + \cdots + r)} = (-1)^{\pi(n+1)}$. When this is done every term becomes positive. Using the above conditions, $k_{jn\cdots r}$ is obtained from $f_{jn\cdots r}$ by making all the signs positive.

## Part II.

### Applications of the formulas to the expansion of special types of products.

Expansions of products having binomial or trinomial factors.

Let $\Pi[1 + t_n x^n]$ be expanded into series. This product is of the general type $\Pi[1 + a_i x + b_i x^2 + c_i x^3 + \cdots\cdots]$ where only one element in each sequence $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\cdots\cdots$ is not zero. Written in full $\Pi[1 + t_n x^n] = [1 + t_1 x + 0 + \cdots][1 + 0 + t_2 x^2 + 0 + \cdots]\cdots\cdots$.
A product of this kind is easily expanded by computing the $C$'s in terms of the $\phi$'s. We have $\phi_1 = t_1$, $\phi_{01} = t_2$, $\phi_{001} = t_3$, $\phi_{11} = t_1 t_2$, $\phi_{101} = t_1 t_3$, $\cdots\cdots$. All the $\phi$'s are zero which have a subscript different from zero or one. Using equations (12)

$$\Pi[1 + t_n x^n] = 1 + t_1 x + t_2 x^2 + (t_1 t_2 + t_3) x^3 + (t_1 t_3 + t_4) x^4 + (t_2 t_3 + t_1 t_4 + t_5) x^5 \cdots.$$

As a special case, in $\Pi[1 + x^n]$ each $t = 1$, and

$$\Pi[1 + x^n] = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots\cdots .$$

As another case, in $\Pi[1 - x^n]$ each $t = -1$, and

$$\Pi[1 - x^n] = 1 - x - x^2 + x^5 + x^7 \cdots\cdots .$$

The product of the type $\Pi[1 + t_n x^{2n}]$ can be expanded by changing $x$ to $x^2$ in the preceding.

Writing out the product $\Pi[1 + t_{2n-1} x^{2n-1}] =$
$[1 + t_1 x + 0 + 0 + \cdots][1 + 0 + C + t_3 x^3 + 0 + \cdots][1 + 0 + 0 + C + C + t_5 x^5 + \cdots]\cdots$
we see that the only $\phi$'s which are not zero are of the form $\phi_{i0j0k0\cdots}$ where $i$, $j$, $k$, $\cdots$ are only zero or one. Using (12)

$$\Pi[1 + t_{2n-1} x^{2n-1}] = 1 + t_1 x + t_3 x^3 + t_1 t_3 x^4 + t_5 x^5 + t_1 t_5 x^6 + t_7 x^7 + (t_1 t_7 + t_3 t_5) x^8 .$$

When all the $t$'s are equal, the preceding expansions are

$$\Pi[1 \pm t x^n] = 1 \pm t x \pm t x^2 + (t^2 \pm t) x^3 + (t^2 \pm t) x^4 + (2t^2 \pm t) x^5 + \cdots\cdots ;$$

$$\Pi[1 \pm t x^{2n}] = 1 \pm t x^2 \pm t x^4 + (t^2 \pm t) x^6 + (t^2 \pm t) x^8 + \cdots\cdots ;$$

$$\Pi[1 \pm t x^{2n-1}] = 1 \pm t x \pm t x^3 + t^2 x^4 \pm t x^5 + t^2 x^6 \pm t x^7 + 2t^2 x^8 + \cdots\cdots .$$

Now let $\Pi[1 + q_n x^n + r_n x^{2n}]$ be expanded into series. This product is of the form $\Pi[1 + a_n x^\alpha + b_n x^\beta + c_n x^\gamma + \cdots\cdots]$ where $a_n = 0$

if $n \neq \alpha$, $b_n = 0$ if $2n \neq \beta$, etc. Written out

$$\Pi = [1 + q_1x + r_1x^2 + 0 + \cdots][1 + 0 + q_2x^2 + 0 + r_2x^4 + 0 + \cdots] \cdots\cdots .$$

The only element in the sequence $\{q_n\}$ whose weight is one, is $q_1$. The only elements of weight two are $q_2$ and $r_1$. In general the only elements of weight $w$ are $q_w$ and $r_{\frac{w}{2}}$ if $\frac{w}{2}$ is an integer. The only terms which go into $C_w$ are products of $q$'s and $r$'s such that their weight is $w$. Then

$$C_w = \Sigma q_{i_1}q_{i_2}\cdots\cdots q_{i_h}r_{j_1}r_{j_2}\cdots r_{j_k} \quad \text{where } \Sigma i + \Sigma 2j = w , \quad \text{and}$$

$$\Pi = 1 + q_1x + (q_2 + r_1)x^2 + (q_3 + q_1q_2)x^3 + (r_2 + q_2r_1 + q_1q_3 + q_4)x^4 + \cdots\cdots .$$

In a similar manner we find, in general,

$$\Pi[1 + q_nx^n + r_nx^{2n} + t_nx^{3n} + \cdots\cdots] =$$

$$1 + q_1x + (q_2 + r_1)x^2 + (q_3 + q_1q_2 + t_1)x^3 + (q_4 + q_1q_3 + r_2 + q_2r_1 + q_1t_1 + u_1)x^4\cdots\cdots$$

in which $C_w = \Sigma q_{i_1}\cdots q_{i_h}r_{j_1}\cdots r_{j_k}t_{l_1}\cdots t_{l_m}\cdots\cdots$

where $\Sigma i + \Sigma 2j + \Sigma 3l \cdots\cdots = w$ .

The product of a finite number of power series may be written out as an application of Theorem II, looking, for convenience, at equations (12) for the expressions of the $C$'s in terms of the $p$'s. As an example, multiply

$$(1 + a_1x + b_1x^2 + \cdots\cdots)(1 + a_2x + b_2x^2 + \cdots\cdots)(1 + a_3x + b_3x^2 + \cdots\cdots) .$$

If the sum of the subscripts of $p_{jk\cdots r}$ is greater than 3, the value of $p$ is zero. Expressed in $a$'s, $b$'s, $c$'s, $\cdots\cdots$ , we have

$$C_1 = a_1 + a_2 + a_3 ,$$

$$C_2 = a_1a_2 + a_1a_3 + a_2a_3 + b_1 + b_2 + b_3 ,$$

$$C_3 = a_1a_2a_3 + a_1b_2 + a_2b_1 + a_1b_3 + a_3b_1 + a_2b_3 + a_3b_2 + c_1 + c_2 + c_3 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

These are the results which would be obtained by the use of Cauchy's rule for multiplying infinite series.

In the case of a finite number of terminating factors, the formulas give the results which would be obtained by multiplying by ordinary algebra. Both the finite and infinite cases are included in the one treatment.

Expansions of the $\vartheta$ and elliptic functions.

In practice, it is often easier in expanding a product of the type of M on page 26, to expand each infinite product into a series and then multiply the series as described above. For example, expand $\vartheta_0(z) = \Pi(1 - q^{2n})\ \Pi(1 - 2q^{2n-1}\cos 2z + q^{4n-2})$. Letting $t^2 = e^{2iz}$ this can be written $\vartheta_0(z) = \Pi(1 - q^{2n})\ \Pi(1 - t^2q^{2n-1})\ \Pi(1 - t^{-2}q^{2n-1})$. From the formulas on page 33

$\Pi(1 - q^{2n}) = 1 - q^2 - q^4 + q^{10} + \cdots\cdots$,

$\Pi(1 - t^2q^{2n-1}) = 1 - t^2q - t^2q^2 + t^4q^4 - t^2q^5 + t^4q^6 - t^2q^7 + 2t^4q^8 + \cdots\cdots$,

$\Pi(1 - t^{-2}q^{2n-1}) = 1 - t^{-2}q - t^{-2}q^2 + t^{-4}q^4 - t^{-2}q^5 + t^{-4}q^6 - t^{-2}q^7 + 2t^{-4}q^8 \cdots\cdots$.

Multiplying these series, using the results of page 34,

$$\vartheta_0(z) = 1 - q(t^2 + t^{-2}) + q^2(t^2t^{-2} - 1) + q^3(t^2 + t^{-2} - t^2 - t^{-2}) + q^4(-t^2t^{-2} + 2t^2t^{-2} - 1 + t^4 + t^{-4}) + \cdots\cdots$$

$$= 1 - 2q\cos 2z + 2q^4\cos 4z - \cdots\cdots$$

In the same manner we obtain

$\vartheta_2(z) = \Pi(1 - q^{2n})\Pi(1 + 2q^{2n-1}\cos 2z + q^{4n-2}) = 1 + 2q\cos 2z + 2q^4\cos 4z + \cdots$,

$\vartheta_1(z) = 2q^{\frac{1}{4}}\sin z\ \Pi(1 - q^{2n})\ \Pi(1 - 2q^{2n}\cos 2z + q^{4n})$

$= q^{\frac{1}{4}}(t - t^{-1})\ \Pi(1 - q^{2n})\ \Pi(1 - t^2q^{2n})\ \Pi(1 - t^{-2}q^{2n})$

$= 2q^{\frac{1}{4}}[\sin z - q^2\sin 3z + q^6\sin 5z \cdots\cdots]$,

$\vartheta_3(z) = 2q^{\frac{1}{4}}\cos z\ \Pi(1 - q^{2n})\ \Pi(1 + 2q^{2n}\cos 2z + q^{4n})$

$= q^{\frac{1}{4}}(t + t^{-1})\ \Pi(1 - q^{2n})\ \Pi(1 + t^2q^{2n})\ \Pi(1 + t^{-2}q^{2n})$

$= 2q^{\frac{1}{4}}[\cos z + q^2\cos 3z + q^6\cos 5z + \cdots\cdots]$.

The product $\Pi(1 \pm 2q^{2n}\cos 2z + q^{4n})$ can be expanded without factoring by using the results of page 34. Let $x = q^2$, $a_r = \pm 2\cos 2z$, and $b_n = 1$. Then

$\Pi(1 \pm 2q^{2n}\cos 2z + q^{4n}) = 1 \pm 2q^2\cos 2z + (1 \pm 2\cos 2z)q^4 + (4\cos^2 2z \pm 2\cos 2z)q^6 \cdots$.

Using this form of the series, the series are obtained in powers of cosines of $2z$, instead of cosines of multiples of $2z$. For example,

$\vartheta_1(z) = 2q^{\frac{1}{4}}\sin z\ [1 - q^2(1 + 2\cos 2z) - q^6(1 + 2\cos 2z - 4\cos^2 2z) \cdots\cdots]$.

$\vartheta_2(z) = 2q^{\frac{1}{4}}\cos z\ [1 - q^2(1 - 2\cos 2z) - q^6(1 + 2\cos 2z - 4\cos^2 2z) \cdots\cdots]$.

As an example of the type shown in Theorem IX, expand

$$\operatorname{sn}\frac{2Kz}{\pi} = \frac{1}{k}\frac{\vartheta_1(z)}{\vartheta_0(z)} = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(z)}{\vartheta_0(z)}$$

$$= \frac{2q^{\frac{1}{4}}\sin z\ \Pi(1-q^{2n})^2\ \Pi(1+2q^{2n-1}+q^{4n-2})\ \Pi(1-2q^{2n}\cos 2z+q^{4n})}{2q^{\frac{1}{4}}\ \Pi(1-q^{2n})^2\ \Pi(1+2q^{2n}+q^{4n})\ \Pi(1-2q^{2n-1}\cos 2z+q^{4n-2})}$$

$$= \frac{(t-t^{-1})\ \Pi(1+q^{2n-1})^2\ \Pi(1-t^2q^{2n})\ \Pi(1-t^{-2}q^{2n})}{2\ \Pi(1+q^{2n})^2\ \Pi(1-t^2q^{2n-1})\ \Pi(1-t^{-2}q^{2n-1})} \quad \text{where } t = e^{iz}.$$

Set $\Pi(1+q^{2n-1}) = (1+q+0+\cdots)(1+0+0+q^6+0+\cdots)(1+\cdots+q^6+\cdots)\cdots$

Here $s'_{i} = 1$, $s'_{0j} = 0$, $s'_{00k} = 1$, $\cdots\cdots$, $s'_{11} = s'_{ij} = 0$.

Set $\Pi(1-t^2q^{2n}) = (1-0-t^2q^2-0-\cdots)(1-0-0-0-t^2q^4-0-\cdots)\cdots\cdots$.

Here $s''_{i} = 0$, $s''_{0j} = (-t^2)^j$, $\cdots\cdots$, $s''_{11} = s''_{ij} = 0$.

For all the separate products we obtain the following s's:

$s'_{i} = 1 = s'_{0\,0\,i} = s'_{0\,0\,0\,0\,i} = \cdots\cdots$

$s''_{0j} = (-t^2)^j = s''_{0\,0\,j} = s''_{0\,0\,0\,0\,0\,j} = \cdots\cdots$

$s'''_{0j} = (-t^{-2})^j = s'''_{0\,0\,j} = s'''_{0\,0\,0\,0\,j} = \cdots\cdots$

$s^{iv}_{0j} = 1 = s^{iv}_{0\,0\,j} = s^{iv}_{0\,0\,0\,0\,j} = \cdots\cdots$

$s^{v}_{i} = (-t^2)^i = s^{v}_{0\,0\,i} = s^{v}_{0\,0\,0\,0\,i} = \cdots\cdots$

$s^{vi}_{i} = (-t^{-2})^i = s^{vi}_{0\,0\,i} = s^{vi}_{0\,0\,0\,0\,i} = \cdots\cdots$ .

In accordance with Theorem IX, let $s = (2s' + s'' + s''') - (2s^{iv} + s^{v} + s^{vi})$.

Using the above table of s's, we find

$s_1 = 2 + t^2 + t^{-2} = 4\cos^2 z = -s_{c1}$

$s_2 = 2 - t^4 - t^{-4} = 4\sin^2 2z = -s_{c2}$

$s_3 = 2 + t^6 + t^{-6} = 4\cos^2 3z = -s_{c3}$

In general, $s_n = -s_{0n} = s_{00n} = -s_{0\,0\,0\,n} = s_{0\,0\,0\,0\,n} = -s_{0\,0\,0\,0\,0\,n} = \cdots\cdots$

and all the other s's are zero.

We also find by direct multiplication that $s_1^2 = 4s_1 - s_2$, $s_1^3 = 15s_1 - 6s_2 + s_3$, $s_1s_2 = s_1 + 2s_2 - s_3$. If we let the C's be related only to the infinite product-quotient without the factor $\frac{t-t^{-1}}{2}$, we obtain

$C_1 = s_1$

$C_2 = \frac{1}{2!}[s_1^2 - s_2 - 2s_1] = s_1 - s_2$

$C_3 = \frac{1}{3!}[s_1^3 - 3s_1s_2 + 2s_3 - 6s_1^2 + 6s_1] = -s_1 - s_2 + s_3$

$C_4 = s_2 + s_3 - s_4$

$C_5 = 2s_1 + s_2 - s_3 - s_4 + s_5$ .

Expressing these $s$'s in terms of the $t$'s and multiplying by $\frac{t - t^{-1}}{2}$, we get

$$\operatorname{sn}\frac{2Kz}{\pi} = q(\sin 3z + \sin z) + q^2(\sin 5z - \sin z) + q^3(\sin 7z - 2\sin 3z - \sin z) + q^4(\sin 9z - 2\sin 5z + \sin 3z + 2\sin z) \cdots\cdots .$$

Expansion of functions requiring a convergence factor; the Gamma function, and the sine function.

It is known that the infinite product $\Pi[(1 + a_n x)e^{Q_{n\nu}(x)}]$ is absolutely-uniformly convergent if $(1 + a_n x)e^{Q_{n\nu}(x)}$ is taken as a single factor and $Q_{n\nu}(x) = -a_n x + \frac{1}{2}a_n^2 x^2 - \cdots + \frac{(-1)^\nu}{\nu} a_n^\nu x^\nu$, and if $\nu$ is chosen so that $\Sigma|a_n|^{\nu+1}$ converges.[26]

By exactly the same reasoning it can be proved that $\Pi[\{1 + v_n(x)\}\, e^{Q_{n\nu}(x)}]$ is absolutely-uniformly convergent if $\{1 + v_n(x)\}\, e^{Q_{n\nu}(x)}$ is taken as a single factor and $Q_{n\nu}(x) = -v_n + \frac{1}{2}v_n^2 \cdots\cdots + \frac{(-1)^\nu}{\nu} v_n^\nu$, and if $\nu$ is chosen so that $\Sigma|v_n|^{\nu+1}$ converges uniformly.

Suppose that the proper choice of $\nu$ has been made. Then $\log\{1 + v_n(x)\}\, e^{Q_{n\nu}(x)}$ has for its *first term*

$$\frac{(-1)^\nu}{\nu + 1}[v_n(x)]^{\nu+1} = \frac{(-1)^\nu}{\nu + 1}(a_n x + b_n x^2 + c_n x^3 + \cdots)^{\nu+1} .$$

When this is expanded by the multinomial theorem (see page 11), every term contains powers of $a_n$, $b_n$, $c_n$, $\cdots$, such that the

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[26] Goursat, Vol. II, p. 127-131.

the total degree of all these elements is $\nu + 1$. Since this is true for each of the factors of the product, when the logs of the factors are added and the $s$'s formed, every $s$ the sum of whose indices is less than $\nu + 1$ will be zero. (See formula (7).)

The weight of $s_{jk\cdots r}$ is $j\alpha + k\beta + \cdots + r\lambda$, $\alpha < \beta < \cdots < \lambda$. If $\alpha = 1$, then the $s$ of lowest weight appearing in the expansion of the product, is $s_{\nu+1}$ and the expansion begins with $1 + s_{\nu+1}x^{\nu+1} \cdots$.

If $\varrho_{n\nu}(x) = -a_n x^\alpha$, then $\log\{[1 + v_n(x)]\, e^{-a_n x^\alpha}\}$ is the same as given on page 11 except that the term $a_n x^\alpha$ does not appear at the beginning. When the sum of the logs is taken, $s_1 x^\alpha$ does not appear. All the other $s$'s appear as given by formulas (7) and (11).

Similarly, if $\varrho_{n\nu}(x) = \dfrac{(-1)^{m+1}\, m!}{j!\,k!\cdots}\, a_n{}^j b_n{}^k \cdots l_n{}^r\, x^h$, where $m = j + k + \cdots + r - 1$ and $h = j\alpha + k\beta + \cdots + r\lambda$, then $s_{jk\cdots r}$ will be zero in the coefficients given by (7) and (11) and no other changes will be made.

As an example of a product of this type, expand $\dfrac{1}{\Gamma(x)} = x\, e^{\gamma x} \Pi[1 + \frac{x}{n}]\, e^{-\frac{x}{n}}$, where $\gamma$ denotes Euler's constant, that is, $\gamma = 0.5772156\cdots$.

Let $\sigma_i = \sum_n \dfrac{1}{n^i}$, $i > 1$.[27] For the infinite product $\Pi[1 + \frac{x}{n}]\, e^{-\frac{x}{n}}$ we have $s_1 = 0$, $s_i = \sigma_i$, $i > 1$. Using equations (12) and expanding $e^{\gamma x}$, we have

$$\frac{1}{\Gamma(x)} = x[1 + \gamma x + \frac{\gamma^2 x^2}{2!} + \frac{\gamma^3 x^3}{3!} + \cdots][1 - \frac{\sigma_2 x^2}{2} + \frac{\sigma_3 x^3}{3} + (-\frac{\sigma_4}{4} + \frac{\sigma_2{}^2}{8})x^4 + \cdots]$$

$$= x[1 + x + (\frac{\gamma^2}{2} - \frac{\sigma_2}{2})x^2 + (\gamma^3 - 3\gamma\sigma_2 + 2\sigma_3)\frac{x^3}{3!} + (\gamma^4 - 6\gamma^2\sigma_2 + 8\gamma\sigma_3 - 6\sigma_4 + 3\sigma_2{}^2)\frac{x^4}{4!}\cdots.$$

It is to be noticed that these coefficients are exactly the $C$'s with $\gamma$ instead of $s_1$, and $\sigma_i$ instead of $s_i$ if $i > 1$.

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[27] For the numerical values of these sums, see Chrystal, Vol.II, p. 367.

Expand $\Gamma(x) = \frac{1}{x} e^{-\gamma x} \{ \Pi [1 + \frac{x}{n}] e^{-\frac{x}{n}} \}^{-1}$ . Using formula (17') for the reciprocal, (page 27), we obtain

$$\Gamma(x) = \frac{1}{x}[1 - \gamma x + \frac{\gamma^2 x^2}{2!} - \frac{\gamma^3 x^3}{3!} + \cdots ][1 + \tfrac{1}{2}\sigma_2 x^2 - \tfrac{1}{3}\sigma_3 x^3 + (\tfrac{1}{4}\sigma_4 + \tfrac{1}{8}\sigma_2^2)x^4 \cdots ]$$

$$= \frac{1}{x} - \gamma + (\gamma^2 + \sigma_2)\frac{x^2}{2} - (\gamma^3 + 3\gamma\sigma_2 + 2\sigma_3)\frac{x^3}{3!}$$

$$+ (\gamma^4 + 6\gamma^2\sigma_2 + 8\gamma\sigma_3 + 6\sigma_4 + 3\sigma_2^2)\frac{x^4}{4!} \cdots\cdots .$$

This result agrees with the rule stated below Theorem IX which, applied to the D function, says that by replacing each $\gamma$ and $\sigma$ by $-\gamma$ and $-\sigma$ respectively, in one series, we obtain the other.[28] This is equivalent to making all the signs positive inside the parentheses in formula (12) and multiplying $C_w$ by $(-1)^w$ . Using the decimal approximations of $\gamma$ and $\sigma$, we get the approximations

$$\Gamma(x) = \frac{1}{x}[\,1 - .5772x + .9891x^2 - 1.8115x^3 + .8928x^4 \cdots\cdots\,] ,$$

$$\frac{1}{\Gamma(x)} = x[\,1 + .5772x - .6559x^2 - .0403x^3 + .2555x^4 \cdots\cdots\,] .$$

Expand the product $\sin x = x \prod_{\infty}^{\infty\,\prime} [1 - \frac{x}{n\pi}]\, e^{\frac{x}{n\pi}}$ . This is absolutely convergent if $e^{\frac{x}{n\pi}}$ is kept with its factor. Then the factors can be paired so that $\sin x = x \prod_{1}^{\infty} [1 - (\frac{x}{n\pi})^2]$ . In this form, $s_j = (-1)^j \frac{(2\pi)^{2j}}{2(2j)!} B_j$ , where $B_j$ is the $j$th Bernoulli number.[29] Replacing each $x$ in the formulas by $\frac{x}{\pi^2}$ and using formulas (11) and (12), we obtain

$$C_1 = -\pi^2 B_1 = -\frac{\pi^2}{3!} , \quad C_2 = \frac{1}{2!}[\pi^4 B_1^2 - \frac{\pi^4 B_2}{3}] = \frac{\pi^4}{5!} , \quad C_3 = -\frac{\pi^6}{7!} , \cdots\cdots .$$

Now $\sin x = x[\,1 + C_1\frac{x^2}{\pi^2} + C_2\frac{x^4}{\pi^4} + C_3\frac{x^6}{\pi^6} + \cdots\cdots\,]$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots .$$

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[28] These results agree with those given by Nielsen, sec. 15.

[29] Price, p. 145. Crelle, Vol. 20, p. 11, Vol. 85, p. 269, for tables of B's.

Expansion of a product in which the $a$'s form a geometric sequence; the O-function.

Let $\Pi = \Pi(1 + a_n x)$ where the $a$'s form a geometric sequence $ar$, $ar^2$, $ar^3$, ····, in which the ratio $r$ may be a complex number[30] but such that $|r| < 1$. This product can be written $\Pi(1 + ar^n x)$, and it is absolutely-uniformly convergent. For this product,

$$s_1 = ar + ar^2 + ar^3 + \cdots\cdots = \frac{ar}{1-r} ,$$

$$s_2 = a^2r^2 + a^2r^4 + a^2r^6 + \cdots\cdots = \frac{a^2r^2}{1-r^2} ,$$

. . . . . .

$$s_n = a^nr^n + a^nr^{2n} + a^nr^{3n} + \cdots\cdots = \frac{a^nr^n}{1-r^n} .$$

Using equations (12) the coefficients of the series are

$$C_1 = \frac{ar}{1-r} , \qquad C_2 = \frac{a^2r \cdot r^2}{(1-r)(1-r^2)} , \qquad \cdots\cdots$$

$$C_n = \frac{a^nr}{(1-r)} \cdot \frac{r^2}{(1-r^2)} \cdot \frac{r^3}{(1-r^3)} \cdots\cdots \frac{r^n}{(1-r^n)} ,$$

and $\Pi = 1 + \Sigma C_n x^n$.

If $x$ in the above product is replaced by $e^{\lambda x}$, then $x^n$ in the series is replaced by $e^{n\lambda x}$ and the coefficients $C_n$ remain unchanged. This product can be written $\Pi[1 + ar^n e^{\lambda x}]$, and it is absolutely-uniformly convergent if $|r| < 1$. This permits us to apply the formulas just given, to another class of products.

As an example, expand the generalized O-function of Heine.[31]

$$O(z) = \Pi\left[1 - e^{(a_1\omega_1 - a_2\omega_2)\frac{\pi i}{\omega_2}} e^{\frac{2\pi i\omega_2}{\omega_1}} e^{\frac{2\pi iz}{\omega_1}}\right] .$$

In this product $a = -e^{(a_1\omega_1 - a_2\omega_2)\frac{\pi i}{\omega_2}}$, $r = e^{\frac{2\pi i\omega_2}{\omega_1}}$, and $\lambda = \frac{2\pi i}{\omega_1}$;

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30 Goursat, sec. 8.

31 This generalization was developed by Professor C. H. Ashton of the University of Kansas. See list of references.

$a_1$ and $a_2$ are arbitrary, and $\omega_1$ and $\omega_2$ may be chosen in any way such that $|r| < 1$. In the usual notation of elliptic functions, this form of $r$ is denoted by $q^2$. The above product becomes

$$\Theta(z) = 1 + \frac{aq}{(1-q)} e^{\lambda z} + \frac{a^2q^3}{(1-q)(1-q^2)} e^{2\lambda z} + \frac{a^3q^6}{(1-q)(1-q^2)(1-q^3)} e^{3\lambda z} \cdots .$$

The examples given in Part II are sufficient to show that the formulas derived in Part I may be used to expand into series a large number of infinite products which have been treated by special methods, provided the product satisfies the conditions of Theorem IV.

## APPENDIX.

Note concerning the weights.

When the number of terms of each factor is finite, some of the numbers $\alpha$, $\beta$, $\gamma$, $\cdots$ may be equal ($\alpha \neq 0$). Requiring each of the $a$'s, $b$'s, $c$'s, etc. to keep its identity without being combined with some other element because of the equalities among the exponents in $v_i(x)$, we see that $p_{jk\cdots r}$ and $s_{jk\cdots r}$ are as defined on pages 2 and 3, and each is of weight $h = j\alpha + k\beta + \cdots + r\lambda$. Theorem I does not depend on the weights and therefore holds true for this case. There is nothing in the proof of Theorem II which is affected by having the weights equal, and therefore Theorem II holds true. Proceeding through the proof of Theorem III, we see that the rearrangement of terms to obtain formula (5) may be performed since $v_i$ is a polynomial and $v_i^m$ therefore satisfies the condition of absolute convergence. Therefore the conditions of the Weierstrass theorem are satisfied and we can select the terms of $u_{ih}$ as before. Now the series $\sum_h u_{ih} x^h = \log [1 + v_i]$ is absolutely convergent, and $\sum_i \log[1 + v_i] = \log \Pi$ is by hypothesis uniformly convergent, so that by the Weierstrass theorem $\sum_i \sum_h u_{ih} x^h = \sum_h \sum_i u_{ih} x^h$; and

$$\sum_i u_{ih} = \sum_i \sum \frac{(-1)^m m!}{j!!} a_i^j b_i^k \cdots l_i^r = \sum \sum_i \frac{(-1)^m m!}{j!!} a_i^j b_i^k \cdots l_i^r = \sum \frac{(-1)^m m!}{j!!} s_{j\cdots r}$$

where $j\alpha + k\beta + \cdots + r\lambda = h$, and $m = j + k + \cdots + r - 1$, and provided the $s$'s converge. This interchange of order of summation may be made because the number of terms in $u_{ih}$ is finite when there is not an infinity of solutions to the equation $j\alpha + k\beta + \cdots + r\lambda = h$. There could not be an infinity of solutions with only a finite number of weights $\alpha$, $\beta$, $\gamma$, $\cdots$, $\lambda$, $\alpha \neq 0$. We have now obtained formula (7) as before, and the remaining portion of the proof follows without changes, and we find that Theorems IIIa, IIIb, and IV as stated will apply to this case.

If some of the weights are equal, some of the $\phi$'s and $s$'s appear in $C$'s of lower weight than they do if the weights are $1, 2, 3, \cdots$. A convenient way of computing the $C$'s in terms of the $s$'s is first to find the $C$'s in terms of the $\phi$'s, and then substitute for each $\phi$ in terms of the $s$'s, since the relations between the $\phi$'s and $s$'s are independent of the weights.

If $\alpha = \beta = \cdots\cdots = \lambda = 1$, then [32]

$$C_1 = \phi_1 + \phi_{01} + \phi_{001} + \cdots\cdots = s_1 + s_{01} + s_{001} + \cdots\cdots;$$

$$C_2 = (\phi_2 + \phi_{02} + \phi_{002} + \cdots) + (\phi_{11} + \phi_{101} + \phi_{011} + \cdots\cdots)$$

$$= \tfrac{1}{2}(s_1^2 + s_{01}^2 + \cdots) - (s_2 + s_{02} + \cdots) - (s_{11} + s_{101} + \cdots) + (s_1 s_{01} + s_1 s_{001} \cdots)$$

$$C_3 = (\phi_3 + \phi_{03} + \cdots) + (\phi_{21} + \phi_{12} + \cdots) + (\phi_{111} + \phi_{1011} + \phi_{0111} + \cdots\cdots).$$

If the product II has only a finite number of factors and each factor is a polynomial, convergence does not need to be considered and all the operations performed in deriving the formulas become algebraic operations. All the theorems and formulas are valid in this special case, but there do not need to be any conditions whatever on the elements of the array (1) or the functions $v_i$.

Note on the choice of notation.

The notation used in this paper cannot be called "standard" for if one reads on the subject of symmetric functions, he will see that no two writers use exactly the same notation; that is, there is no standard. Careful consideration has been given to the choice of notation used, an effort being made to select symbols which agree with as many writers as possible. Only a few writers have considered more than one sequence of elements, but nevertheless, there are differences in the use of the subscripts attached to the letters denoting functions. For example, compare Junker with Macmahon. I do not maintain that my notation is the best that can be devised, but I believe it is systematic and readable.

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[32] See Cullis, Vol. III, Chap. 3, and Macmahon, Vol. II, Sec. XI.

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This list includes works on algebra and analysis which have been useful in preparing this paper. A few are listed here which have not been referred to specifically. It is not a complete bibliography of all works on symmetric functions.

## Vita.

Charles Arthur Messick was born in Wilson, Kansas, in 1896, but at the age of two moved with his parents to Abiline, Kansas, where his education began. At the age of nine he moved with his parents to Topeka, Kansas, and there he graduated from the grade schools, and from Topeka High School with honors, in 1915. He attended Washburn College at Topeka, and Kansas State Teacher's College at Emporia from which he received the B. S. degree in the summer of 1918. He taught four years in Kansas high schools, after which he went to the University of Kansas and became Assistant Instructor in the Department of Mathematics in 1923. In 1925 he received the M. A. degree in Mathematics, from the University of Kansas, having taken courses with Professors C. H. Ashton, U. G. Mitchell, and S. Lefschetz. He then became a full time Assistant in the Department of Mathematics at the University of Iowa, remaining there until 1927, and studying under Professor E. W. Chittenden in addition to his regular teaching schedule. From 1927 to the time of this writing he has been Assistant Professor of Mathematics at the University of Florida. Every summer beginning with the summer of 1925 and continuing through the summer of 1930 he has studied at the University of Chicago in the Department of Mathematics, taking courses taught by Professors H. E. Slaught, G. A. Bliss, E. P. Lane, E. T. Bell, Dunham Jackson, and others. The writer wishes to express his sincere appreciation of the valuable instruction and assistance given by all the above Professors, and most especially to E. T. Bell and L. W. Graves, who have supervised the writing of this dissertation. Sincere thanks are expressed to them for their valuable comments and suggestions.